Introduction to Nonlinear Control

Stability, control design, and estimation

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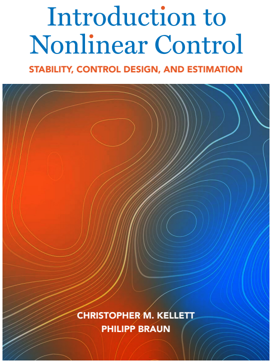
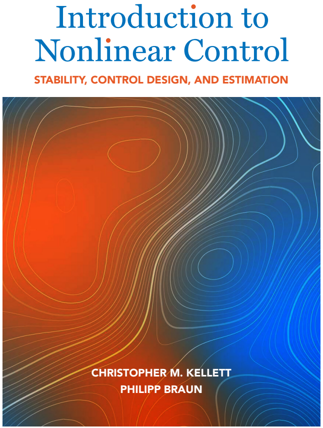
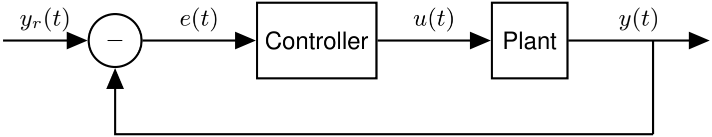




Part II: Controller Design

Output Regulation



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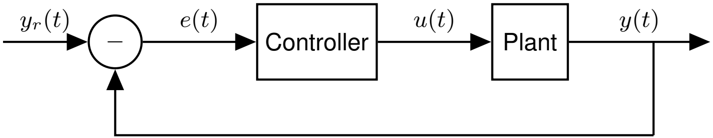
$$\dot{x} = Ax + Bu + E_d d$$
$$y = Cx + Du + F_d d,$$

with disturbance d

Output Regulation



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Goal: asymptotically track a reference signal

$y(t) \to y_r(t)$ (regardless of the disturbance)

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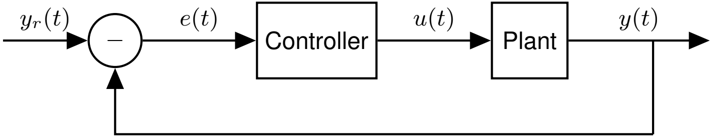

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$$\begin{aligned} e(t) &= y(t) - y_r(t) \\ &= Cx(t) + Du(t) + F_d d(t) - y_r(t) \end{aligned}$$

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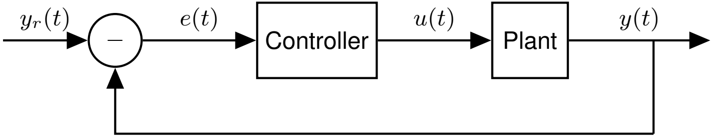

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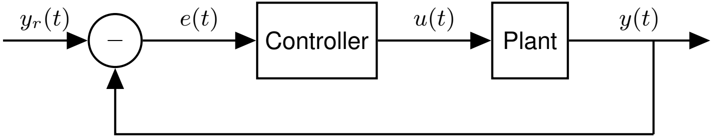

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Additional Assumptions/Definitions:

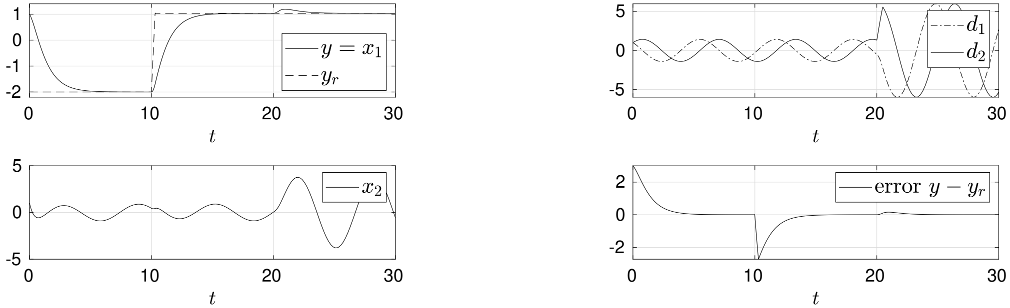
- Exosystem:

$$w = \begin{bmatrix} y_r \\ d \end{bmatrix}, \quad \dot{w} = A_1 w, \quad A_1 = \begin{bmatrix} A_r & 0 \\ 0 & A_d \end{bmatrix}.$$

- Overall system dynamics:

$$\dot{x} = Ax + Bu + [0 \;\; E_d]w = Ax + Bu + Ew$$
$$\dot{w} = A_1 w$$
$$e = Cx + Du + [-I \;\; F_d]w = Cx + Du + Fw$$

Example



Setting and Observations:

- Tracking of a piecewise constant reference signal
- The frequency of the disturbance d changes after 20 seconds
- Only convergence of the output but not the state is guaranteed

Output Regulation: Problem Formulation and Controller design

(Linear) Output regulation problem: For

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew \\ \dot{w} &= A_1 w \\ e &= Cx + Du + Fw \end{aligned}$$

define

$$u = K_x x + K_w w$$

such that

- $\lim_{t\to\infty} e(t) = 0$
- $\lim_{t\to\infty} x(t) = 0$ for $w \equiv 0$.

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It holds that

$$\begin{aligned}\dot{x}_c &= (A + BK_x)x_c + (E + BK_w)w = A_c x_c + B_c w \\ \dot{w} &= A_1 w \\ e &= (C + DK_x)x_c + (F + DK_w)w = C_c x_c + D_c w\end{aligned}$$

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Task: Find K_x and K_w such that

- A_c is Hurwitz and
- $\lim_{t\to\infty} e(t) = \lim_{t\to\infty} (C_c x_c(t) + D_c w(t)) = 0$

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Lemma

Assume A_c is Hurwitz and A_1 has no eigenvalues with negative real parts. Then $\lim_{t\to\infty} e(t) = 0$ iff there exists a unique X_c satisfying

$$X_c A_1 = A_c X_c + B_c, \qquad 0 = C_c X_c + D_c.$$

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1. Select K_x so that $A_c = A + BK_x$ is Hurwitz.
2. Solve (unknowns: X_c and K_w)

$$X_c A_1 = (A + BK_x)\, X_c + BK_w + E$$
$$0 = (C + DK_x)\, X_c + DK_w + F$$

Output Regulation: Motivation and Problem Formulation

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Consider coordinate transformation:

$$\begin{bmatrix} X \\ U \end{bmatrix} = \begin{bmatrix} I & 0 \\ K_x & I \end{bmatrix} \begin{bmatrix} X_c \\ K_w \end{bmatrix}$$

leading to (*regulator equations*)

$$\begin{aligned} XA_1 &= AX + BU + E \\ 0 &= CX + DU + F \end{aligned}$$

- linear in unknowns X, U
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Theorem (Regulator equations)

The regulator equations are solvable for any matrices E and F iff for all eigenvalues λ of A_1, it holds that

$$\operatorname{rank}\left(\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}\right) = n + p$$

(where the matrix has dimension $(n+p) \times (n+p)$)

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Extension 2: Nonlinear Output Regulation:

$$\dot{x} = F(x, u, w)$$
$$\dot{w} = a_1(w)$$
$$e = H(x, u, w)$$

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