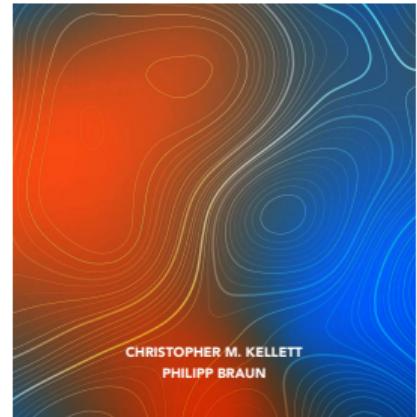


Introduction to Nonlinear Control

Stability, control design, and estimation

Christopher M. Kellett & Philipp Braun

Introduction to
Nonlinear Control
STABILITY, CONTROL DESIGN, AND ESTIMATION



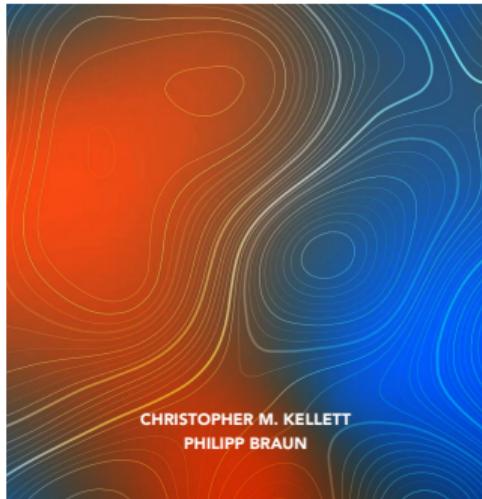
Part II: Controller Design

13 Output Regulation

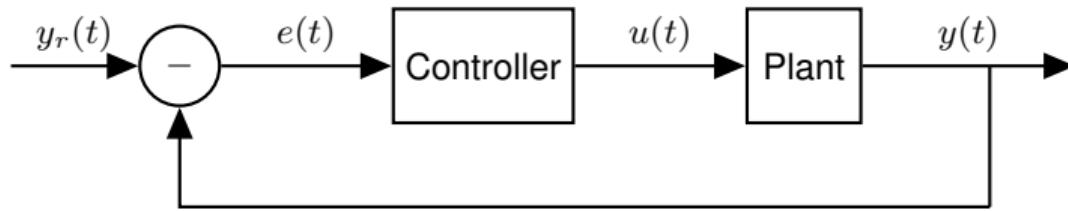
- 13.1 Linear Output Regulation
- 13.2 Robust Linear Output Regulation
- 13.3 Nonlinear Output Regulation
- 13.4 Exercises
- 13.5 Bibliographical Notes and Further Reading

Introduction to Nonlinear Control

STABILITY, CONTROL DESIGN, AND ESTIMATION



Output Regulation



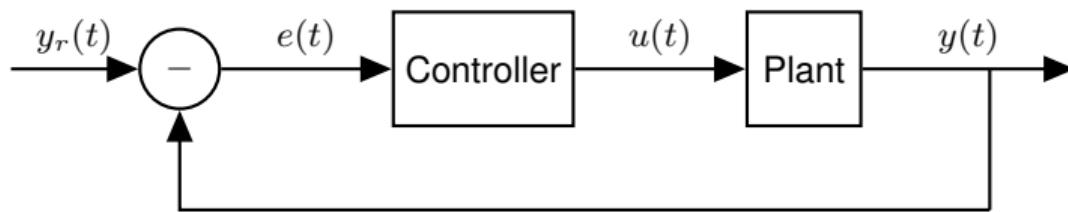
Consider perturbed linear system:

$$\dot{x} = Ax + Bu + E_d d$$

$$y = Cx + Du + F_d d,$$

with disturbance d

Output Regulation



Goal: asymptotically track a reference signal

$y(t) \rightarrow y_r(t)$ (regardless of the disturbance)

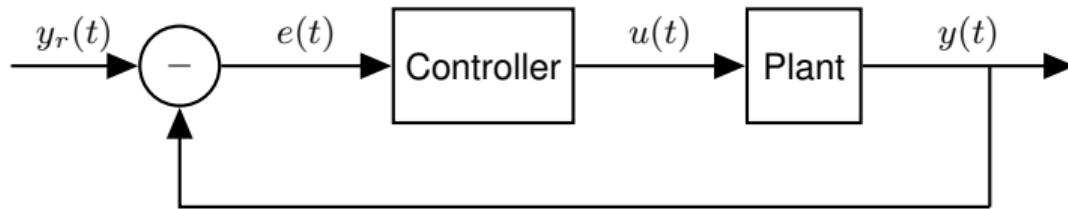
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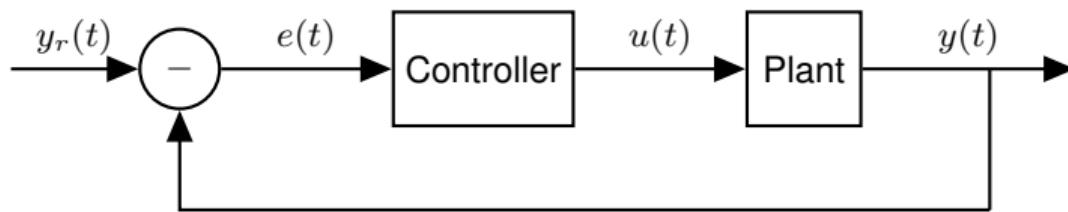
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$$y(t) \rightarrow y_r(t) \quad (\text{regardless of the disturbance})$$

Tracking error:

$$\begin{aligned} e(t) &= y(t) - y_r(t) \\ &= Cx(t) + Du(t) + F_d d(t) - y_r(t) \end{aligned}$$

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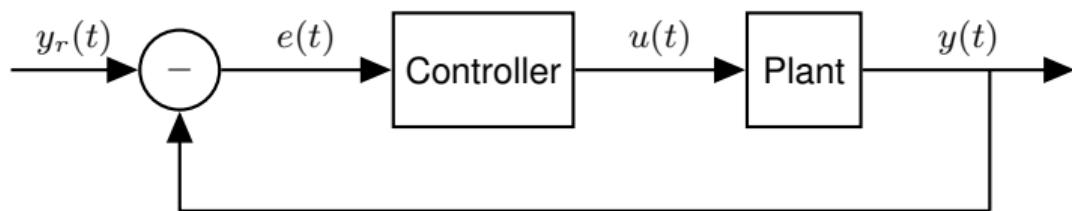
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Additional Assumptions/Definitions:

- Exosystem:

$$w = \begin{bmatrix} y_r \\ d \end{bmatrix}, \quad \dot{w} = A_1 w, \quad A_1 = \begin{bmatrix} A_r & 0 \\ 0 & A_d \end{bmatrix}.$$

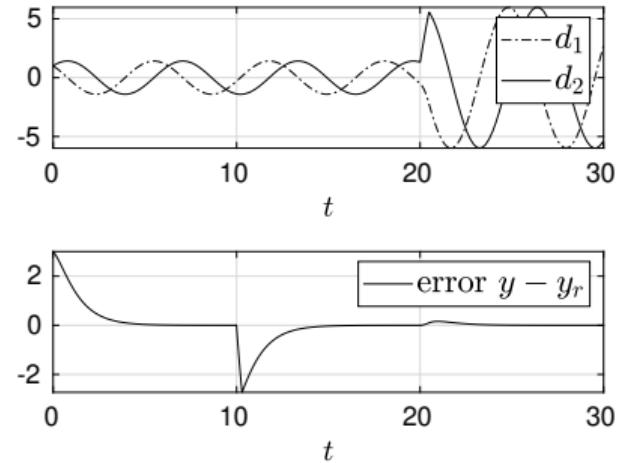
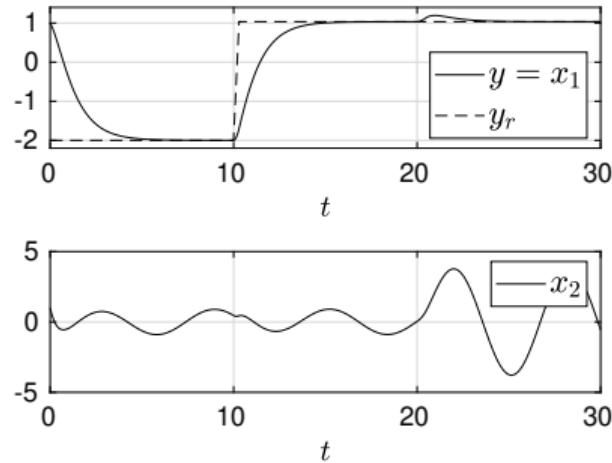
- Overall system dynamics:

$$\dot{x} = Ax + Bu + [0 \ E_d]w = Ax + Bu + Ew$$

$$\dot{w} = A_1 w$$

$$e = Cx + Du + [-I \ F_d]w = Cx + Du + Fw$$

Example



Setting and Observations:

- Tracking of a piecewise constant reference signal
- The frequency of the disturbance d changes after 20 seconds
- Only convergence of the output but not the state is guaranteed

Output Regulation: Problem Formulation and Controller design

(Linear) Output regulation problem: For

$$\dot{x} = Ax + Bu + Ew$$

$$\dot{w} = A_1 w$$

$$e = Cx + Du + Fw$$

define

$$u = K_x x + K_w w$$

such that

- $\lim_{t \rightarrow \infty} e(t) = 0$
- $\lim_{t \rightarrow \infty} x(t) = 0$ for $w \equiv 0$.

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It holds that

$$\dot{x}_c = (A + BK_x)x_c + (E + BK_w)w = A_c x_c + B_c w$$

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$$e = (C + DK_x)x_c + (F + DK_w)w = C_c x_c + D_c w$$

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Task: Find K_x and K_w such that

- A_c is Hurwitz and
- $\lim_{t \rightarrow \infty} e(t) = \lim_{t \rightarrow \infty} (C_c x_c(t) + D_c w(t)) = 0$

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Lemma

Assume A_c is Hurwitz and A_1 has no eigenvalues with negative real parts. Then $\lim_{t \rightarrow \infty} e(t) = 0$ iff there exists a unique X_c satisfying

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- 1 Select K_x so that $A_c = A + BK_x$ is Hurwitz.
- 2 Solve (unknowns: X_c and K_w)

$$\begin{aligned} X_c A_1 &= (A + BK_x) X_c + BK_w + E \\ 0 &= (C + DK_x) X_c + DK_w + F \end{aligned}$$

Output Regulation: Motivation and Problem Formulation

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Consider coordinate transformation:

$$\begin{bmatrix} X \\ U \end{bmatrix} = \begin{bmatrix} I & 0 \\ K_x & I \end{bmatrix} \begin{bmatrix} X_c \\ K_w \end{bmatrix}$$

leading to (*regulator equations*)

$$\begin{aligned} X A_1 &= A X + B U + E \\ 0 &= C X + D U + F \end{aligned}$$

- linear in unknowns X, U
- $K_w = U - K_x X$ for all K_x

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Theorem (Regulator equations)

The *regulator equations* are solvable for any matrices E and F iff for all eigenvalues λ of A_1 , it holds that

$$\text{rank} \left(\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \right) = n + p$$

(where the matrix has dimension $(n + p) \times (n + p)$)

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Extension 1: Uncertain linear systems

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Extension 2: Nonlinear Output Regulation:

$$\dot{x} = F(x, u, w)$$

$$\dot{w} = a_1(w)$$

$$e = H(x, u, w)$$

Introduction to Nonlinear Control: Stability, control design, and estimation

Part I: Dynamical Systems

1. Nonlinear Systems - Fundamentals & Examples
2. Nonlinear Systems - Stability Notions
3. Linear Systems and Linearization
4. Frequency Domain Analysis
5. Discrete Time Systems
6. Absolute Stability
7. Input-to-State Stability

Part II: Controller Design

8. LMI Based Controller and Antiwindup Designs
9. Control Lyapunov Functions
10. Sliding Mode Control
11. Adaptive Control
12. Introduction to Differential Geometric Methods
13. Output Regulation
14. Optimal Control
15. Model Predictive Control

Part III: Observer Design & Estimation

16. Observer Design for Linear Systems
17. Extended & Unscented Kalman Filter & Moving Horizon Estimation
18. Observer Design for Nonlinear Systems