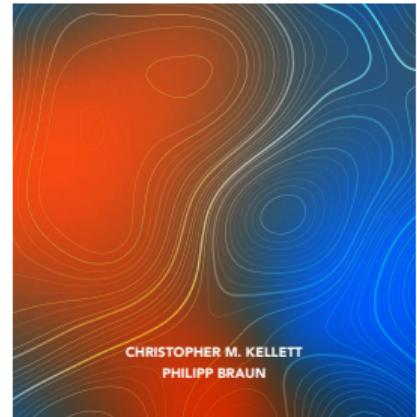


Introduction to Nonlinear Control

Stability, control design, and estimation

Christopher M. Kellett & Philipp Braun

Introduction to
Nonlinear Control
STABILITY, CONTROL DESIGN, AND ESTIMATION



Part II: Controller Design

12 Introduction to Differential Geometric Methods

12.1 Introductory Examples

12.2 Zero Dynamics and Relative Degree

12.3 Feedback Linearization

12.3.1 Nonlinear Controllability

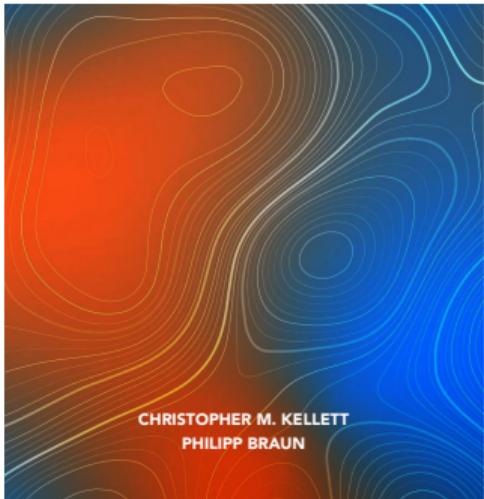
12.3.2 Input-to-State Linearization

12.4 Exercises

12.5 Bibliographical Notes and Further Reading

Introduction to Nonlinear Control

STABILITY, CONTROL DESIGN, AND ESTIMATION



Introductory Examples: (Example 1)

Consider (nonlinear system)

$$\dot{x} = x^3 + u, \quad y = x.$$

Goal: Stabilize the state/output $y = x = 0$.

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Solution: Linear Controller Design

- Linearization about the origin:

$$\dot{x} = u$$

- Natural stabilizing controller selection

$$u = -kx \quad (k > 0)$$

- Nonlinear closed loop dynamics:

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Nonlinear Controller Design:

- Consider the nonlinear feedback

$$u = -x^3 + v, \quad v \text{ to be designed}$$

- Closed-loop system $\dot{x} = v$

- Natural feedback selection

$$v = -kx \quad (k > 0)$$

- Closed-loop system:

$$\dot{x} = -kx \quad (\text{with globally as. stable origin})$$

- Overall feedback law:

$$u = -x^3 - kx$$

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Note that:

- Coordinate transformation leads to a linear system
- Global instead of local asymptotic stability

Introductory Examples: (Example 2)

Consider second-order system

$$\dot{x}_1 = x_2 + x_1^2$$

$$\dot{x}_2 = -2x_1^3 - 2x_1x_2 + u$$

$$y = x_1$$

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Consider change of coordinates

$$z_1 = x_1$$

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System in new coordinates:

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$$y = z_1,$$

Note that: The system is linear in z !

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Linear/Nonlinear feedback law ($k_1, k_2 > 0$):

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$$= -k_1 x_1 - k_2 (x_2 + x_1^2)$$

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Global asymptotic stability of the origin can be easily verified by checking the eigenvalues of the linear closed-loop system.

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Global asymptotic stability of the origin can be easily verified by checking the eigenvalues of the linear closed-loop system.

Note that:

- Coordinate transformation allows us to stabilize and analyze a linear system instead of a nonlinear system.
- $x \rightarrow 0$ is equivalent to $z \rightarrow 0$.
- For the input-output behavior it is not important if the dynamics are written in terms of x or z .

Introductory Examples: (Example 3)

Consider the nonlinear system:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3^3 + u \quad y = x_1, \\ \dot{x}_3 &= x_1 + x_3^3\end{aligned}$$

Consider change of coordinates:

$$\begin{aligned}z_1 &= x_3 \\ z_2 &= x_1 + x_3^3 \\ z_3 &= x_2 + 3x_1x_3^2 + 3x_3^5.\end{aligned}$$

and initial feedback (with v to be designed):

$$u = -x_3^3 - 3x_2x_3^2 - 6x_1^2x_3 - 21x_1x_3^4 - 15x_3^7 + v,$$

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Leads to linear states (but a nonlinear output):

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \quad y = z_2 - z_1^3 \\ \dot{z}_3 &= v\end{aligned}\tag{1}$$

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Here,

- we are able to partially linearize the dynamics
- the “internal” (nonlinear) x_3 dynamics, are not visible through the output
- for (1), v can be defined such that the origin $z = 0$ is asymptotically stable (i.e., y converges to zero).
- for (2) a controller guaranteeing $y(t) \rightarrow 0$ for $t \rightarrow \infty$ can be defined using pole placement. (But is the origin asymptotically stable?)

In this chapter we will discuss . . .

Feedback linearization

- Input-to-state linearization
- Input-to-output linearization

Relies on properties such as

- relative degree
- zero dynamics

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- (repeated) Lie derivatives
 $(\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m, f : \mathbb{R}^n \rightarrow \mathbb{R}^n)$

$$L_f^0 \lambda(x) = \lambda(x)$$

$$L_f \lambda(x) = \frac{\partial \lambda}{\partial x}(x) \cdot f(x)$$

$$L_f^k \lambda(x) = \frac{\partial}{\partial x} \left(L_f^{k-1} \lambda(x) \right) f(x),$$

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- Input-to-output linearization

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This also allows us to talk about

- nonlinear controllability (accessibility)

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Introduction to Nonlinear Control: Stability, control design, and estimation

Part I: Dynamical Systems

1. Nonlinear Systems - Fundamentals & Examples
2. Nonlinear Systems - Stability Notions
3. Linear Systems and Linearization
4. Frequency Domain Analysis
5. Discrete Time Systems
6. Absolute Stability
7. Input-to-State Stability

Part II: Controller Design

8. LMI Based Controller and Antiwindup Designs
9. Control Lyapunov Functions
10. Sliding Mode Control
11. Adaptive Control
12. Introduction to Differential Geometric Methods
13. Output Regulation
14. Optimal Control
15. Model Predictive Control

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16. Observer Design for Linear Systems
17. Extended & Unscented Kalman Filter & Moving Horizon Estimation
18. Observer Design for Nonlinear Systems