Introduction to Nonlinear Control

Stability, control design, and estimation

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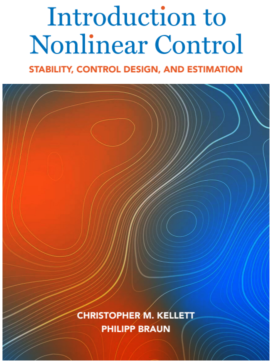

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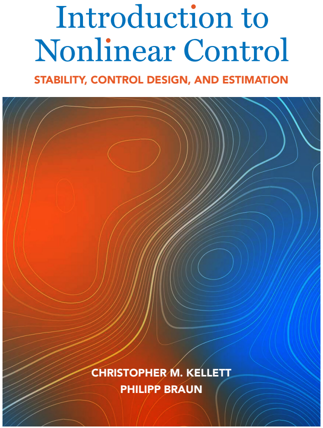
Part II: Controller Design



Introductory Examples: (Example 1)

Consider (nonlinear system)

$$\dot{x} = x^3 + u, \quad y = x.$$

Goal: Stabilize the state/output $y = x = 0$.

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Solution: Linear Controller Design

- Linearization about the origin:

$$\dot{x} = u$$

- Natural stabilizing controller selection

$$u = -kx \qquad (k > 0)$$

- Nonlinear closed loop dynamics:

$$\dot{x} = x(x^2 - k), \quad x^e \in \{0, \pm\sqrt{k}\}.$$

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- Region of attraction increases with k.

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Nonlinear Controller Design:

- Consider the nonlinear feedback

$$u = -x^3 + v, \quad v \text{ to be designed}$$

- Closed-loop system $\quad \dot{x} = v$
- Natural feedback selection

$$v = -kx \qquad (k > 0)$$

- Closed-loop system:

$$\dot{x} = -kx \qquad \text{(with globally as. stable origin)}$$

- Overall feedback law:

$$u = -x^3 - kx$$

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Note that:

- Coordinate transformation leads to a linear system
- Global instead of local asymptotic stability

Introductory Examples: (Example 2)

Consider second-order system

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1^2 \\ \dot{x}_2 &= -2x_1^3 - 2x_1 x_2 + u \\ y &= x_1 \end{aligned}$$

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System in new coordinates:

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= \dot{x}_2 + 2x_1\dot{x}_1 = u \\ y &= z_1,\end{aligned}$$

Note that: The system is linear in z!

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Linear/Nonlinear feedback law $(k_1, k_2 > 0)$:

$$\begin{aligned}u &= -k_1 z_1 - k_2 z_2\\ &= -k_1 x_1 - k_2\left(x_2 + x_1^2\right)\end{aligned}$$

Global asymptotic stability of the origin can be easily verified by checking the eigenvalues of the linear closed-loop system.

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Global asymptotic stability of the origin can be easily verified by checking the eigenvalues of the linear closed-loop system.

Note that:

- Coordinate transformation allows us to stabilize and analyze a linear system instead of a nonlinear system.
- $x \to 0$ is equivalent to $z \to 0$.
- For the input-output behavior it is not important if the dynamics are written in terms of x or z.

Introductory Examples: (Example 3)

Consider the nonlinear system:

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Consider change of coordinates:

$$\begin{aligned} z_1 &= x_3 \\ z_2 &= x_1 + x_3^3 \\ z_3 &= x_2 + 3x_1x_3^2 + 3x_3^5. \end{aligned}$$

and initial feedback (with v to be designed):

$$u = -x_3^3 - 3x_2x_3^2 - 6x_1^2x_3 - 21x_1x_3^4 - 15x_3^7 + v,$$

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Leads to linear states (but a nonlinear output):

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The feedback law

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Leads to linear input-output relationship from v to y:

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Here,

- we are able to partially linearize the dynamics
- the “internal” (nonlinear) x_3 dynamics, are not visible through the output
- for (1), v can be defined such that the origin $z = 0$ is asymptotically stable (i.e., y converges to zero).
- for (2) a controller guaranteeing $y(t) \to 0$ for $t \to \infty$ can be defined using pole placement. (But is the origin asymptotically stable?)

In this chapter we will discuss . . .

Feedback linearization

- Input-to-state linearization
- Input-to-output linearization

Relies on properties such as

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- (repeated) Lie derivatives
  ($\lambda : \mathbb{R}^n \to \mathbb{R}^m$, $f : \mathbb{R}^n \to \mathbb{R}^n$)

$$L_f^0 \lambda(x) = \lambda(x)$$
$$L_f \lambda(x) = \frac{\partial \lambda}{\partial x}(x) \cdot f(x)$$
$$L_f^k \lambda(x) = \frac{\partial}{\partial x} \left(L_f^{k-1} \lambda(x) \right) f(x),$$

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This also allows us to talk about

- nonlinear controllability (accessibility)

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