

# Introduction to Nonlinear Control

Stability, control design, and estimation

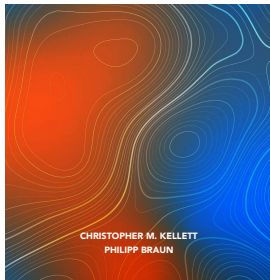
Christopher M. Kellett & Philipp Braun



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STABILITY, CONTROL DESIGN, AND ESTIMATION

CHRISTOPHER M. KELLETT  
PHILIPP BRAUN



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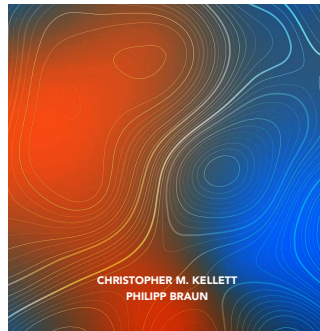
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- Consider  $V(x) = \frac{1}{2}x^2$  which satisfies

$$\begin{aligned}\dot{V}(x) &= -k_1x^2 - k_2x^4 + \theta x^2 \\ &\leq -k_1x^2 - (k_2 - \tfrac{1}{2})x^4 + \tfrac{1}{2}\theta^2,\end{aligned}$$

thus it holds that

$$x(t) \xrightarrow{t \rightarrow \infty} S_\theta = \left\{ x \in \mathbb{R} \mid |x| \leq \sqrt{\frac{1}{k_1}|\theta|} \right\}$$



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**We can conclude that**

- **Bound on  $\theta$  known:** Global asymptotic stability of 0 can be guaranteed ( $k_1 > \theta$ )
- **Bound on  $\theta$  not known:** Convergence to neighborhood around 0 can be guaranteed

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$$V(x, \hat{\theta}) = \frac{1}{2}x^2 + \frac{1}{2}\hat{\theta}^2$$

which satisfies

$$\begin{aligned} \dot{V}(x, \hat{\theta}) &= (-\hat{\theta} x - k_1 x)x + \hat{\theta} x^2 \\ &= -k_1 x^2 \end{aligned}$$

- Then LaSalle-Yoshizawa theorem implies that

- ▶  $x(t) \xrightarrow{t \rightarrow \infty} 0$  for all  $(x_0, \xi_0) \in \mathbb{R}^2$
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Dynamic controller designs can be used to guarantee global convergence properties!

# Model Reference Adaptive Control

- Consider linear systems

$$\dot{x} = Ax + Bu$$

with unknown matrices  $A, B$ .

- Goal:** Design a controller so that the unknown system behaves like

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u^e$$

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- Overall system dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{\bar{x}} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} (A + BL(\theta))x + BM(\theta)u^e \\ \bar{A}\bar{x} + \bar{B}u^e \\ \Psi(x, \bar{x}, u^e) \end{bmatrix}$$

for  $\Psi$  defined appropriately so that  $x(t) \rightarrow \bar{x}(t)$

# Adaptive Backstepping (for Nonlinear Dynamics)

Systems in *parametric strict-feedback form*:

$$\dot{x}_1 = x_2 + \phi_1(x_1)^T \theta$$

$$\dot{x}_2 = x_3 + \phi_2(x_1, x_2)^T \theta$$

$$\vdots$$

$$\dot{x}_{n-1} = x_n + \phi_{n-1}(x_1, \dots, x_{n-1})^T \theta$$

$$\dot{x}_n = \beta(x)u + \phi_n(x)^T \theta$$

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## Theorem

Let  $c_i > 0$  for  $i \in \{1, \dots, n\}$ . Consider the adaptive controller

$$u = \frac{1}{\beta(x)} \alpha_n(x, \vartheta_1, \dots, \vartheta_n)$$

$$\dot{\vartheta}_i = \Gamma \left( \phi_i(x_1, \dots, x_i) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \phi_j(x_1, \dots, x_j) \right) z_i, \quad i = 1, \dots, n,$$

where  $\vartheta_i \in \mathbb{R}^q$  are multiple estimates of  $\theta$ ,  $\Gamma > 0$  is the adaptation gain matrix, and the variables  $z_i$  and the stabilizing functions

$$\alpha_i = \alpha_i(x_1, \dots, x_i, \vartheta_1, \dots, \vartheta_i), \quad \alpha_i : \mathbb{R}^{i+q} \rightarrow \mathbb{R}, \quad i = 1, \dots, n,$$

are defined by the following recursive expressions (and  $z_0 \equiv 0$ ,  $\alpha_0 \equiv 0$  for notational convenience)

$$z_i = x_i - \alpha_{i-1}(x_1, \dots, x_i, \vartheta_1, \dots, \vartheta_i)$$

$$\begin{aligned}\alpha_i &= -c_i z_i - z_{i-1} - \left( \phi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \phi_j \right)^T \vartheta_i \\ &\quad + \sum_{j=1}^{i-1} \left( \frac{\partial \alpha_{i-1}}{\partial x_j} x_{j+1} + \frac{\partial \alpha_{i-1}}{\partial \vartheta_j} \Gamma \left( \phi_j - \sum_{k=1}^{j-1} \frac{\partial \alpha_{j-1}}{\partial x_k} \phi_k \right) z_j \right).\end{aligned}$$

This adaptive controller guarantees global boundedness of  $x(\cdot)$ ,  $\vartheta_1(\cdot)$ ,  $\dots$ ,  $\vartheta_n(\cdot)$ , and  $x_1(t) \rightarrow 0$ ,  $x_i(t) \rightarrow x_i^e$  for  $i = 2, \dots, n$  for  $t \rightarrow \infty$  where

$$x_i^e = -\theta^T \phi_{i-1}(0, x_2^e, \dots, x_{i-1}^e), \quad i = 2, \dots, n.$$

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## Part I: Dynamical Systems

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2. Nonlinear Systems - Stability Notions
3. Linear Systems and Linearization
4. Frequency Domain Analysis
5. Discrete Time Systems
6. Absolute Stability
7. Input-to-State Stability

## Part II: Controller Design

8. LMI Based Controller and Antiwindup Designs
9. Control Lyapunov Functions
10. Sliding Mode Control
11. Adaptive Control
12. Introduction to Differential Geometric Methods
13. Output Regulation
14. Optimal Control
15. Model Predictive Control

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17. Extended & Unscented Kalman Filter & Moving Horizon Estimation
18. Observer Design for Nonlinear Systems