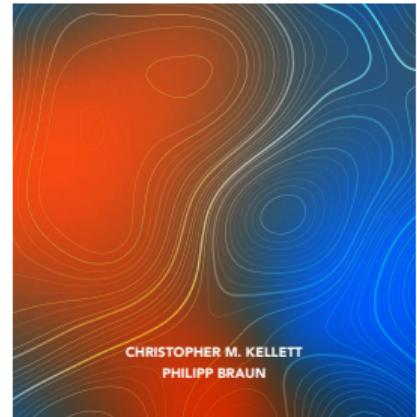


# Introduction to Nonlinear Control

Stability, control design, and estimation

Christopher M. Kellett & Philipp Braun

Introduction to  
Nonlinear Control  
STABILITY, CONTROL DESIGN, AND ESTIMATION



## Part II: Controller Design

### 11 Adaptive Control

#### 11.1 Motivating Examples and Challenges

- 11.1.1 Limitations of Static Feedback Laws
- 11.1.2 Estimation-Based Controller Designs

#### 11.2 Model Reference Adaptive Control

#### 11.3 Adaptive Control for Nonlinear Systems

- 11.3.1 Adaptive Backstepping

- 11.3.2 Tuning Function Designs

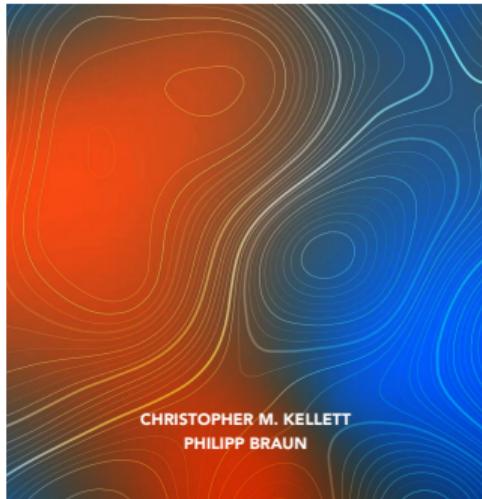
- 11.3.3 Application: Single Link Manipulator with Flexible Joint

#### 11.4 Exercises

#### 11.5 Bibliographical Notes and Further Reading

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$$\begin{aligned}\dot{V}(x) &= -k_1x^2 - k_2x^4 + \theta x^2 \\ &\leq -k_1x^2 - (k_2 - \frac{1}{2})x^4 + \frac{1}{2}\theta^2,\end{aligned}$$

thus it holds that

$$x(t) \xrightarrow{t \rightarrow \infty} S_\theta = \left\{ x \in \mathbb{R} \mid |x| \leq \sqrt{\frac{1}{k_1}|\theta|} \right\}$$

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We can conclude that

- **Bound on  $\theta$  known:** Global asymptotic stability of 0 can be guaranteed ( $k_1 > \theta$ )
- **Bound on  $\theta$  not known:** Convergence to neighborhood around 0 can be guaranteed

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which satisfies

$$\begin{aligned} \dot{V}(x, \hat{\theta}) &= (-\hat{\theta} x - k_1 x)x + \hat{\theta} x^2 \\ &= -k_1 x^2 \end{aligned}$$

- Then LaSalle-Yoshizawa theorem implies that
  - ▶  $x(t) \xrightarrow{t \rightarrow \infty} 0$  for all  $(x_0, \xi_0) \in \mathbb{R}^2$
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Dynamic controller designs can be used to guarantee global convergence properties!

# Model Reference Adaptive Control

- Consider linear systems

$$\dot{x} = Ax + Bu$$

with unknown matrices  $A, B$ .

- Goal:** Design a controller so that the unknown system behaves like

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u^e$$

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- Overall system dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{\bar{x}} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} (A + BL(\theta))x + BM(\theta)u^e \\ A\bar{x} + Bu^e \\ \Psi(x, \bar{x}, u^e) \end{bmatrix}$$

for  $\Psi$  defined appropriately so that  $x(t) \rightarrow \bar{x}(t)$

Systems in *parametric strict-feedback form*:

$$\dot{x}_1 = x_2 + \phi_1(x_1)^T \theta$$

$$\dot{x}_2 = x_3 + \phi_2(x_1, x_2)^T \theta$$

⋮

$$\dot{x}_{n-1} = x_n + \phi_{n-1}(x_1, \dots, x_{n-1})^T \theta$$

$$\dot{x}_n = \beta(x)u + \phi_n(x)^T \theta$$

where  $\beta(x) \neq 0$  for all  $x \in \mathbb{R}^n$

# Adaptive Backstepping (for Nonlinear Dynamics)

## Theorem

Let  $c_i > 0$  for  $i \in \{1, \dots, n\}$ . Consider the adaptive controller

$$u = \frac{1}{\beta(x)} \alpha_n(x, \vartheta_1, \dots, \vartheta_n)$$

$$\dot{v}_i = \Gamma \left( \phi_i(x_1, \dots, x_i) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \phi_j(x_1, \dots, x_j) \right) z_i, \quad i = 1, \dots, n,$$

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where  $\vartheta_i \in \mathbb{R}^q$  are multiple estimates of  $\theta$ ,  $\Gamma > 0$  is the adaptation gain matrix, and the variables  $z_i$  and the stabilizing functions

$$\alpha_i = \alpha_i(x_1, \dots, x_i, \vartheta_1, \dots, \vartheta_i), \quad \alpha_i : \mathbb{R}^{i+i \cdot q} \rightarrow \mathbb{R}, \quad i = 1, \dots, n,$$

are defined by the following recursive expressions (and  $z_0 \equiv 0$ ,  $\alpha_0 \equiv 0$  for notational convenience)

$$z_i = x_i - \alpha_{i-1}(x_1, \dots, x_i, \vartheta_1, \dots, \vartheta_i)$$

$$\begin{aligned} \alpha_i &= -c_i z_i - z_{i-1} - \left( \phi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \phi_j \right)^T \vartheta_i \\ &\quad + \sum_{j=1}^{i-1} \left( \frac{\partial \alpha_{i-1}}{\partial x_j} x_{j+1} + \frac{\partial \alpha_{i-1}}{\partial \vartheta_j} \Gamma \left( \phi_j - \sum_{k=1}^{j-1} \frac{\partial \alpha_{j-1}}{\partial x_k} \phi_k \right) z_j \right). \end{aligned}$$

This adaptive controller guarantees global boundedness of  $x(\cdot)$ ,  $\vartheta_1(\cdot)$ ,  $\dots$ ,  $\vartheta_n(\cdot)$ , and  $x_1(t) \rightarrow 0$ ,  $x_i(t) \rightarrow x_i^e$  for  $i = 2, \dots, n$  for  $t \rightarrow \infty$  where

$$x_i^e = -\theta^T \phi_{i-1}(0, x_2^e, \dots, x_{i-1}^e), \quad i = 2, \dots, n.$$

# Introduction to Nonlinear Control: Stability, control design, and estimation

## Part I: Dynamical Systems

1. Nonlinear Systems - Fundamentals & Examples
2. Nonlinear Systems - Stability Notions
3. Linear Systems and Linearization
4. Frequency Domain Analysis
5. Discrete Time Systems
6. Absolute Stability
7. Input-to-State Stability

## Part II: Controller Design

8. LMI Based Controller and Antiwindup Designs
9. Control Lyapunov Functions
10. Sliding Mode Control
11. Adaptive Control
12. Introduction to Differential Geometric Methods
13. Output Regulation
14. Optimal Control
15. Model Predictive Control

## Part III: Observer Design & Estimation

16. Observer Design for Linear Systems
17. Extended & Unscented Kalman Filter & Moving Horizon Estimation
18. Observer Design for Nonlinear Systems