# Introduction to Nonlinear Control

## Stability, control design, and estimation

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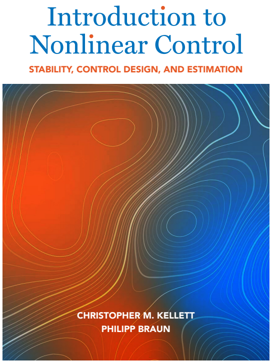
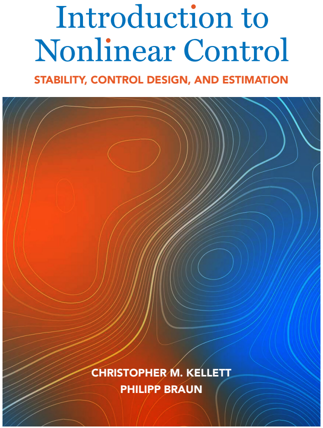


## Part II: Controller Design





# Motivations and Examples

Consider a parameter-dependent system:

$$\dot{x} = f(x, u, \theta), \qquad (\theta \in \mathbb{R}^q \text{ constant but unknown})$$

Goal: Stabilization of the origin.

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- Linear controller: For $u = -kx$ it holds that

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- Consider $V(x) = \frac{1}{2}x^2$ which satisfies

  $$\begin{aligned}\dot{V}(x) &= -k_1 x^2 - k_2 x^4 + \theta x^2 \\ &\leq -k_1 x^2 - (k_2 - \tfrac{1}{2})x^4 + \tfrac{1}{2}\theta^2,\end{aligned}$$

  thus it holds that

  $$x(t) \overset{t\to\infty}{\rightarrow} S_\theta = \left\{ x \in \mathbb{R} \;\middle|\; |x| \leq \sqrt{\tfrac{1}{k_1}}|\theta| \right\}$$

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We can conclude that

- Bound on $\theta$ known: Global asymptotic stability of $0$ can be guaranteed $(k_1 > \theta)$
- Bound on $\theta$ not known: Convergence to neighborhood around $0$ can be guaranteed

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$$\left[\begin{array}{c} \dot{x} \\ \hline \dot{\xi} \end{array}\right] = \left[\begin{array}{c} \theta x - k_1 x - \xi x \\ \hline x^2 \end{array}\right]$$

- and in terms of error dynamics: $\hat{\theta} = \xi - \theta$

$$\left[\begin{array}{c} \dot{x} \\ \hline \dot{\hat{\theta}} \end{array}\right] = \left[\begin{array}{c} -\hat{\theta} x - k_1 x \\ \hline x^2 \end{array}\right]$$

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- Consider candidate Lyapunov function

$$V(x, \hat{\theta}) = \frac{1}{2}x^2 + \frac{1}{2}\hat{\theta}^2$$

which satisfies

$$\begin{aligned} \dot{V}(x, \hat{\theta}) &= (-\hat{\theta} x - k_1 x)x + \hat{\theta} x^2 \\ &= -k_1 x^2 \end{aligned}$$

- Then LaSalle-Yoshizawa theorem implies that
  - $x(t) \overset{t\to\infty}{\to} 0$ for all $(x_0, \xi_0) \in \mathbb{R}^2$
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> Dynamic controller designs can be used to guarantee global convergence properties!

# Model Reference Adaptive Control

- Consider linear systems

$$\dot{x} = Ax + Bu$$

with unknown matrices $A$, $B$.

- Goal: Design a controller so that the unknown system behaves like

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u^e$$

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- Closed-loop dynamics:
$$\begin{aligned}\dot{x} &= Ax + B(M(\theta)u^e + L(\theta)x) \\ &= (A + BL(\theta))x + BM(\theta)u^e \\ &= A_{\mathrm{cl}}(\theta)x + B_{\mathrm{cl}}(\theta)u^e\end{aligned}$$
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- Compatibility conditions

$$\begin{aligned}A_{\mathrm{cl}}(\theta) = \bar{A} \qquad &\Longleftrightarrow \qquad BL(\theta) = \bar{A} - A, \\ B_{\mathrm{cl}}(\theta) = \bar{B} \qquad &\Longleftrightarrow \qquad BM(\theta) = \bar{B}.\end{aligned}$$

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- Overall system dynamics

$$\begin{bmatrix} \dot{x} \\ \hline \dot{\bar{x}} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} (A + BL(\theta))x + BM(\theta)u^e \\ \hline \bar{A}\bar{x} + \bar{B}u^e \\ \Psi(x, \bar{x}, u^e) \end{bmatrix}$$

for $\Psi$ defined appropriately so that $x(t) \to \bar{x}(t)$

# Adaptive Backstepping (for Nonlinear Dynamics)

Systems in *parametric strict-feedback form*:

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \phi_1(x_1)^T\theta \\
\dot{x}_2 &= x_3 + \phi_2(x_1, x_2)^T\theta \\
&\vdots \\
\dot{x}_{n-1} &= x_n + \phi_{n-1}(x_1, \dots, x_{n-1})^T\theta \\
\dot{x}_n &= \beta(x)u + \phi_n(x)^T\theta
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## Theorem

*Let* $c_i > 0$ *for* $i \in \{1, \ldots, n\}$. *Consider the adaptive controller*

$$u = \frac{1}{\beta(x)}\alpha_n(x, \vartheta_1, \ldots, \vartheta_n)$$

$$\dot{\vartheta}_i = \Gamma\left(\phi_i(x_1, \ldots, x_i) - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\phi_j(x_1, \ldots x_j)\right)z_i, \quad i = 1, \ldots, n,$$

*where* $\vartheta_i \in \mathbb{R}^q$ *are multiple estimates of* $\theta$, $\Gamma > 0$ *is the adaptation gain matrix, and the variables* $z_i$ *and the stabilizing functions*

$$\alpha_i = \alpha_i(x_1, \ldots, x_i, \vartheta_1, \ldots, \vartheta_i), \qquad \alpha_i : \mathbb{R}^{i+i\cdot q} \to \mathbb{R}, \qquad i = 1, \ldots, n,$$

*are defined by the following recursive expressions (and* $z_0 \equiv 0$, $\alpha_0 \equiv 0$ *for notational convenience)*

$$z_i = x_i - \alpha_{i-1}(x_1, \ldots, x_i, \vartheta_1, \ldots, \vartheta_i)$$

$$\begin{aligned}
\alpha_i = -c_i z_i - z_{i-1} - \left(\phi_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\phi_j\right)^T\vartheta_i \\
+ \sum_{j=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_j}x_{j+1} + \frac{\partial\alpha_{i-1}}{\partial\vartheta_j}\Gamma\left(\phi_j - \sum_{k=1}^{j-1}\frac{\partial\alpha_{j-1}}{\partial x_k}\phi_k\right)z_j\right).
\end{aligned}$$

*This adaptive controller guarantees global boundedness of* $x(\cdot)$, $\vartheta_1(\cdot)$, $\ldots$, $\vartheta_n(\cdot)$, *and* $x_1(t) \to 0$, $x_i(t) \to x_i^e$ *for* $i = 2, \ldots, n$ *for* $t \to \infty$ *where*

$$x_i^e = -\theta^T\phi_{i-1}(0, x_2^e, \ldots, x_{i-1}^e), \qquad i = 2, \ldots, n.$$

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