Introduction to Nonlinear Control

Stability, control design, and estimation

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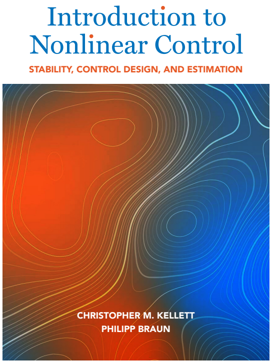
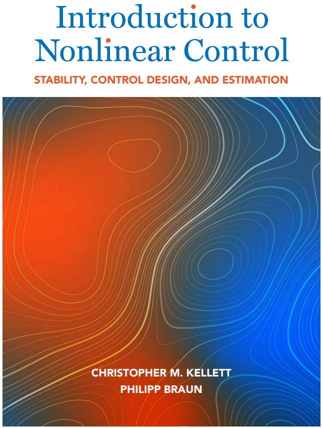


Part II: Controller Design



Setting & Motivation: Sliding Mode Controller Design

We consider systems of the form

$$\dot{x} = f(x, u, \delta(t, x))$$
$$y = h(x)$$

with

- state $x \in \mathbb{R}^n$
- input $u \in \mathbb{R}^m$
- output $y \in \mathbb{R}$
- potentially time and state dependent unknown disturbance $\delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$

We will be interested in

- stabilizing the origin

despite the presence of the disturbance.

$\rightsquigarrow$ First we have to discuss *finite-time stability*.

From Asymptotic Stability . . .

Definition (Asymptotic Stability)

Consider $\dot{x} = f(x)$ with $f(0) = 0$.

- The origin is *(Lyapunov) stable* if, for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if

$$|x(0)| \leq \delta \quad \text{implies} \quad |x(t)| \leq \varepsilon \quad \forall\, t \geq 0.$$

- The origin is *attractive* if there exists $\delta > 0$ such that if $|x(0)| < \delta$ then

$$\lim_{t\to\infty} x(t) = 0.$$

- The origin is *asymptotically stable* for $\dot{x} = f(x)$ if it is both stable and attractive.

Theorem (Asymptotic stability theorem)

Suppose there exist $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{P}$ such that, for all $x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$\langle \nabla V(x), f(x)\rangle \leq -\rho(|x|)$$

Then the origin is (globally) asymptotically stable.

. . . to Finite-Time Stability

Definition (Finite-time stability)

Consider $\dot{x} = f(x)$ with $f(0) = 0$.
The origin is globally finite-time stable if there exists $T : \mathbb{R}^n \backslash \{0\} \to (0, \infty)$, called the settling-time function, such that the following hold:

- (Stability)
  $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that, $x(0) \in \mathcal{B}_\delta \backslash \{0\}$ implies
  $$x(t) \in \mathcal{B}_\varepsilon \qquad \forall\, t \in [0, T(x_0))$$
- (Finite-time convergence)
  $\forall\, x(0) \in \mathbb{R}^n \backslash \{0\}$,
  - $x(\cdot)$ is defined on $[0, T(x_0))$,
  - $x(t) \in \mathbb{R}^n \backslash \{0\}$ for all $t \in [0, T(x_0))$
  - $x(t) \to 0$ for $t \to T(x_0)$.

Theorem (Lyapunov fcn for finite-time stability)

Assume there exist a continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, which is continuously differentiable on $\mathbb{R}^n \backslash \{0\}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a constant $\kappa > 0$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|),$$
$$\dot{V}(x) = \langle \nabla V(x), f(x) \rangle \leq -\kappa \sqrt{V(x)} \qquad \forall x \neq 0.$$

Then the origin is globally finite-time stable.

Moreover, the settling-time $T(x) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is upper bounded by

$$T(x) \leq \tfrac{2}{\kappa} \sqrt{\alpha_2(|x|)}.$$

Finite-Time Stability (Example)

Example

Consider

$$\dot{x} = f(x) = -\operatorname{sign}(x)\sqrt[3]{x^2}.$$

We can verify

$$x(t) = \begin{cases} -\frac{1}{27}\operatorname{sign}(x(0))(t - 3\sqrt[3]{|x(0)|})^3 & \text{if } t \leq 3\sqrt[3]{|x(0)|} \\ 0 & \text{if } t \geq 3\sqrt[3]{|x(0)|} \end{cases}$$

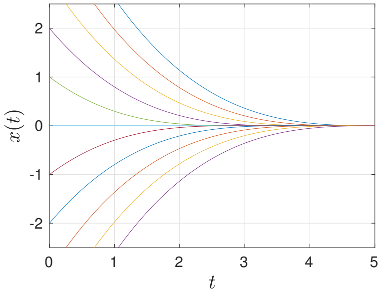
Once the equilibrium is reached, the inequalities

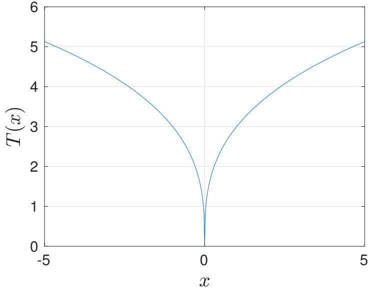
$$-\operatorname{sign}(x)\sqrt[3]{x^2} < 0 \text{ for all } x > 0, \quad \text{and}$$

$$-\operatorname{sign}(x)\sqrt[3]{x^2} > 0 \text{ for all } x < 0$$

ensure that the origin is attractive.
One can show that

- The origin is finite-time stable (with Lyapunov fcn $V(x) = \sqrt[3]{x^2}$)
- Settling time $T(x) = 3\sqrt[3]{|x|}$





Basic Sliding Mode Control

As an example, consider:

$$\dot{x} = x^3 + z,$$
$$\dot{z} = u + \delta(t, x, z).$$

- Unknown disturbance $\delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \to \mathbb{R}$
- Assumption: there exists $L_\delta \in \mathbb{R}_{>0}$ such that

$$|\delta(t, x, z)| \leq L_\delta \qquad (t, x, z) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2$$

- Thus, δ is bounded but not necessarily continuous

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Goal: Exponential stability of the x-subsystem

- I.e., we want x to behave as $\dot{x} = -x$ (for all bounded disturbances)
- The desired behavior implies $\dot{x} + x = 0$
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$$x^3 + z + x = 0$$

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Approach: Define a new state

$$\sigma \doteq x^3 + z + x \quad \text{and} \quad V(\sigma) = \tfrac{1}{2}\sigma^2$$

- Then

$$\begin{aligned} \dot{V}(\sigma) &= \sigma\dot{\sigma} = \sigma\left(3x^2\dot{x} + \dot{z} + \dot{x}\right) \\ &= \sigma\left(3x^5 + 3x^2 z + u + \delta(t, x, z) + x^3 + z\right). \end{aligned}$$

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- To cancel the known terms define

$$u = v - 3x^5 - 3x^2 z - x^3 - z$$

so that $\quad \dot{V}(\sigma) = \sigma\left(v + \delta(t, x, z)\right) \quad$ (with new input v)

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$$\dot{V}(\sigma) = \sigma\left(-\rho\ \mathrm{sign}(\sigma) + \delta(t, x, z)\right) = -\rho|\sigma| + \sigma\delta(t, x, z)$$
$$\leq -\rho|\sigma| + L_\delta|\sigma| = -(\rho - L_\delta)|\sigma|.$$

- Finally, with $\rho = L_\delta + \frac{\kappa}{\sqrt{2}}$, $\kappa > 0$, we have

$$\dot{V}(\sigma) \leq -\frac{\kappa|\sigma|}{\sqrt{2}} = -\alpha\sqrt{V(\sigma)}$$ $\rightsquigarrow$ finite-time stab. of $\sigma = 0$

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- Note that the control

$$u = -\left(L_\delta + \tfrac{\kappa}{\sqrt{2}}\right) \operatorname{sign}\left(x^3 + z + x\right) - 3x^5 - 3x^2 z - x^3 - z$$

is independent of the term $\delta(t, x, z)$.

Basic Sliding Mode Control – Explicit Example

Consider:

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Control law:

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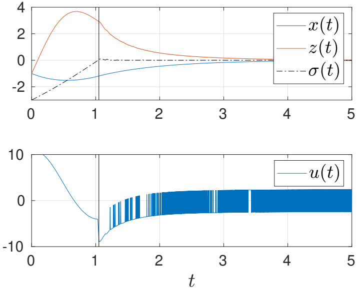
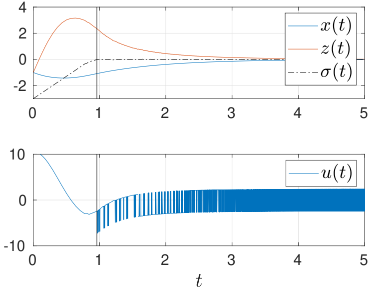
Parameter selection for the simulations:

- $L_\delta = 1$ and $\kappa = 2$
- $\delta(t, x, z) = \sin(t)$ (top)
- $\delta(t, x, z) = \operatorname{sign}(\cos(2t)\sin(2t))$ (bottom)





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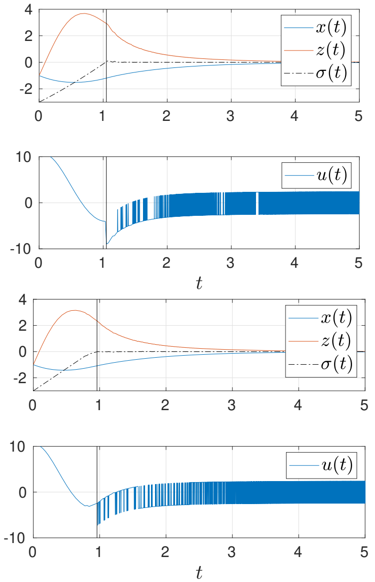
- σ converges to zero in finite-time
- Afterwards (x, z) asymptotically approach the origin
- Since the ordinary differential equation is solved numerically, σ is not exactly zero!



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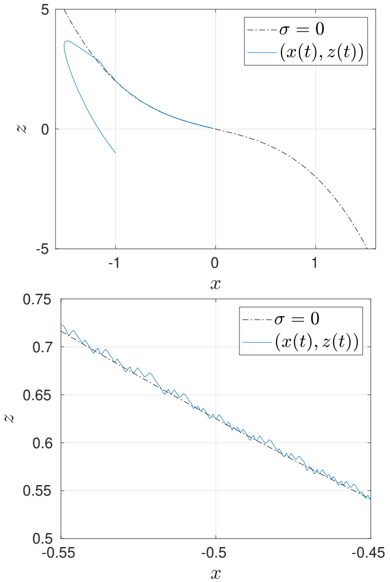
Convergence structure:

$\rightsquigarrow$ Similar to backstepping/forwarding



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