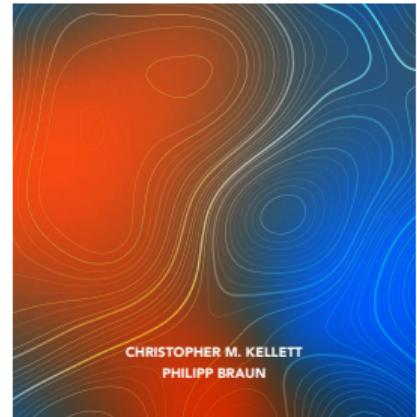


# Introduction to Nonlinear Control

Stability, control design, and estimation

Christopher M. Kellett & Philipp Braun

Introduction to  
Nonlinear Control  
STABILITY, CONTROL DESIGN, AND ESTIMATION



## Part II: Controller Design

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10.2.2 Chattering and Chattering Avoidance

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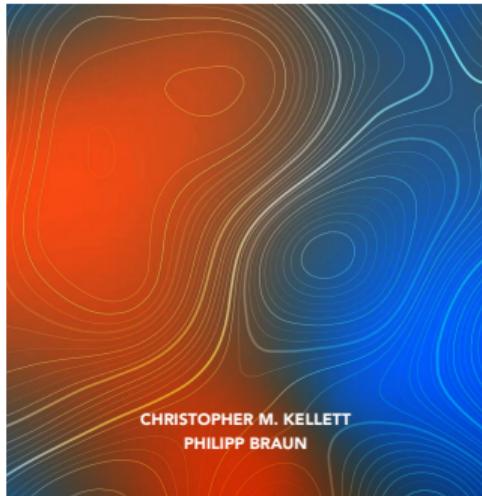
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# Introduction to Nonlinear Control

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## Setting & Motivation: Sliding Mode Controller Design

We consider systems of the form

$$\dot{x} = f(x, u, \delta(t, x))$$

$$y = h(x)$$

with

- state  $x \in \mathbb{R}^n$
- input  $u \in \mathbb{R}^m$
- output  $y \in \mathbb{R}$
- potentially time and state dependent unknown disturbance  $\delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$

We will be interested in

- stabilizing the origin

despite the presence of the disturbance.

~ First we have to discuss *finite-time stability*.

## Definition (Asymptotic Stability)

Consider  $\dot{x} = f(x)$  with  $f(0) = 0$ .

- The origin is *(Lyapunov) stable* if, for any  $\varepsilon > 0$  there exists  $\delta = \delta(\varepsilon) > 0$  such that if

$$|x(0)| \leq \delta \quad \text{implies} \quad |x(t)| \leq \varepsilon \quad \forall t \geq 0.$$

- The origin is *attractive* if there exists  $\delta > 0$  such that if  $|x(0)| < \delta$  then

$$\lim_{t \rightarrow \infty} x(t) = 0.$$

- The origin is *asymptotically stable* for  $\dot{x} = f(x)$  if it is both **stable and attractive**.

## Theorem (Asymptotic stability theorem)

Suppose there exist  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ ,  $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$  and  $\rho \in \mathcal{P}$  such that, **for all**  $x \in \mathbb{R}^n$ ,

$$\begin{aligned} \alpha_1(|x|) &\leq V(x) \leq \alpha_2(|x|) \\ \langle \nabla V(x), f(x) \rangle &\leq -\rho(|x|) \end{aligned}$$

*Then the origin is (globally) asymptotically stable.*

## Definition (Finite-time stability)

Consider  $\dot{x} = f(x)$  with  $f(0) = 0$ .

The origin is globally **finite-time stable** if there exists  $T : \mathbb{R}^n \setminus \{0\} \rightarrow (0, \infty)$ , called the **settling-time function**, such that the following hold:

- **(Stability)**

$\forall \varepsilon > 0 \exists \delta > 0$  such that,  $x(0) \in \mathcal{B}_\delta \setminus \{0\}$  implies  
 $x(t) \in \mathcal{B}_\varepsilon \quad \forall t \in [0, T(x_0))$

- **(Finite-time convergence)**

$\forall x(0) \in \mathbb{R}^n \setminus \{0\}$ ,

- ▶  $x(\cdot)$  is defined on  $[0, T(x_0))$ ,
- ▶  $x(t) \in \mathbb{R}^n \setminus \{0\}$  for all  $t \in [0, T(x_0))$
- ▶  $x(t) \rightarrow 0$  for  $t \rightarrow T(x_0)$ .

## Theorem (Lyapunov fcn for finite-time stability)

Assume there exist a **continuous function**  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ , which is continuously differentiable on  $\mathbb{R}^n \setminus \{0\}$ ,  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  and a constant  $\kappa > 0$  such that

$$\begin{aligned}\alpha_1(|x|) &\leq V(x) \leq \alpha_2(|x|), \\ \dot{V}(x) = \langle \nabla V(x), f(x) \rangle &\leq -\kappa \sqrt{V(x)} \quad \forall x \neq 0.\end{aligned}$$

*Then the origin is globally finite-time stable.*

Moreover, the **settling-time**  $T(x) : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is upper bounded by

$$T(x) \leq \frac{2}{\kappa} \sqrt{\alpha_2(|x|)}.$$

# Finite-Time Stability (Example)

## Example

Consider

$$\dot{x} = f(x) = -\text{sign}(x) \sqrt[3]{x^2}.$$

We can verify

$$x(t) = \begin{cases} -\frac{1}{27} \text{sign}(x(0))(t - 3\sqrt[3]{|x(0)|})^3 & \text{if } t \leq 3\sqrt[3]{|x(0)|} \\ 0 & \text{if } t \geq 3\sqrt[3]{|x(0)|} \end{cases}$$

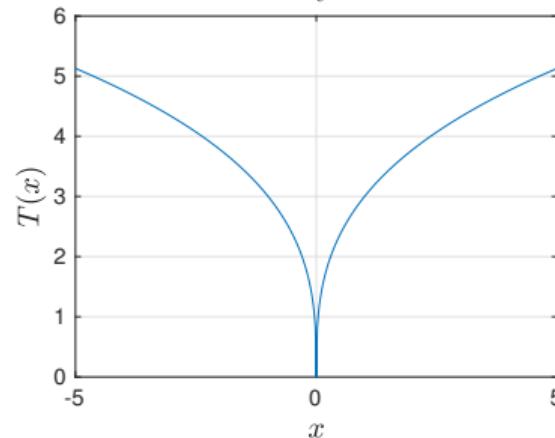
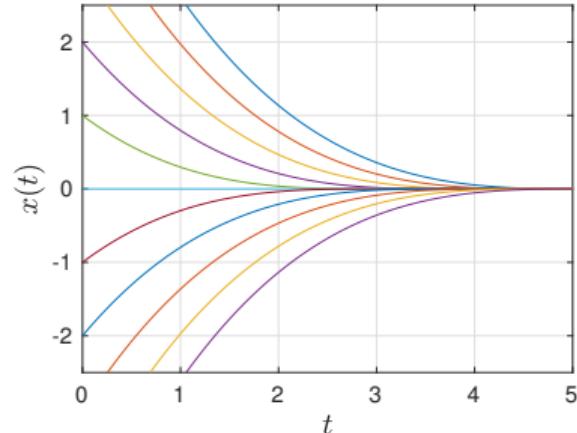
Once the equilibrium is reached, the inequalities

- $\text{sign}(x) \sqrt[3]{x^2} < 0$  for all  $x > 0$ , and
- $\text{sign}(x) \sqrt[3]{x^2} > 0$  for all  $x < 0$

ensure that the origin is attractive.

One can show that

- The origin is finite-time stable (with Lyapunov fcn  $V(x) = \sqrt[3]{x^2}$ )
- Settling time  $T(x) = 3\sqrt[3]{|x|}$



# Basic Sliding Mode Control

As an example, consider:

$$\begin{aligned}\dot{x} &= x^3 + z, \\ \dot{z} &= u + \delta(t, x, z).\end{aligned}$$

- Unknown disturbance  $\delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \rightarrow \mathbb{R}$
- Assumption: there exists  $L_\delta \in \mathbb{R}_{>0}$  such that
$$|\delta(t, x, z)| \leq L_\delta \quad (t, x, z) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2$$
- Thus,  $\delta$  is bounded but not necessarily continuous

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Goal: Exponential stability of the  $x$ -subsystem

- I.e., we want  $x$  to behave as  $\dot{x} = -x$  (for all bounded disturbances)
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Approach: Define a new state

$$\sigma \doteq x^3 + z + x \quad \text{and} \quad V(\sigma) = \frac{1}{2}\sigma^2$$

- Then

$$\begin{aligned}\dot{V}(\sigma) &= \sigma \dot{\sigma} = \sigma (3x^2 \dot{x} + \dot{z} + \dot{x}) \\ &= \sigma (3x^5 + 3x^2 z + u + \delta(t, x, z) + x^3 + z).\end{aligned}$$

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$$u = v - 3x^5 - 3x^2 z - x^3 - z$$

so that  $\dot{V}(\sigma) = \sigma (v + \delta(t, x, z))$  (with new input  $v$ )

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- Selecting  $v = -\rho \operatorname{sign}(\sigma)$ ,  $\rho > 0$ , provides the estimate
- $$\begin{aligned}\dot{V}(\sigma) &= \sigma (-\rho \operatorname{sign}(\sigma) + \delta(t, x, z)) = -\rho |\sigma| + \sigma \delta(t, x, z) \\ &\leq -\rho |\sigma| + L_\delta |\sigma| = -(\rho - L_\delta) |\sigma|.\end{aligned}$$
- Finally, with  $\rho = L_\delta + \frac{\kappa}{\sqrt{2}}$ ,  $\kappa > 0$ , we have

$$\dot{V}(\sigma) \leq -\frac{\kappa |\sigma|}{\sqrt{2}} = -\alpha \sqrt{V(\sigma)} \rightsquigarrow \text{finite-time stab. of } \sigma = 0$$

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- Note that the control

$$u = -\left(L_\delta + \frac{\kappa}{\sqrt{2}}\right) \operatorname{sign}(x^3 + z + x) - 3x^5 - 3x^2 z - x^3 - z$$

is independent of the term  $\delta(t, x, z)$ .

# Basic Sliding Mode Control – Explicit Example

Consider:

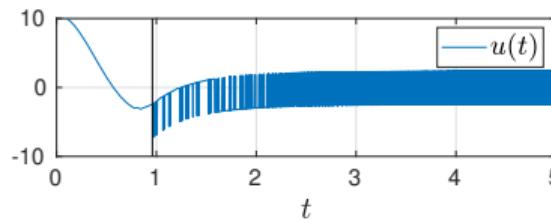
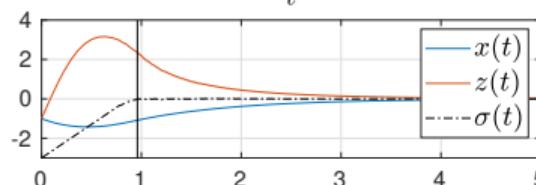
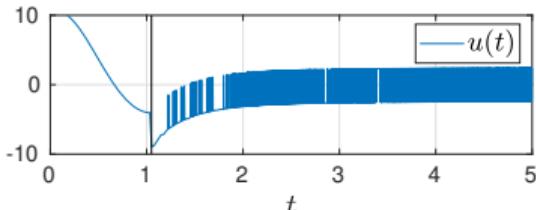
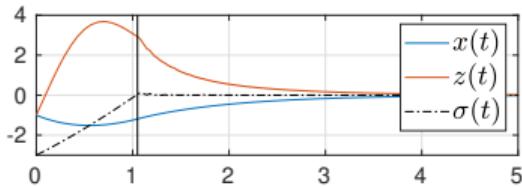
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Control law:

$$u = - \left( L_\delta + \frac{\kappa}{\sqrt{2}} \right) \operatorname{sign}(x^3 + z + x) - 3x^5 - 3x^2 z - x^3 - z$$

Parameter selection for the simulations:

- $L_\delta = 1$  and  $\kappa = 2$
- $\delta(t, x, z) = \sin(t)$  (top)
- $\delta(t, x, z) = \operatorname{sign}(\cos(2t) \sin(2t))$  (bottom)



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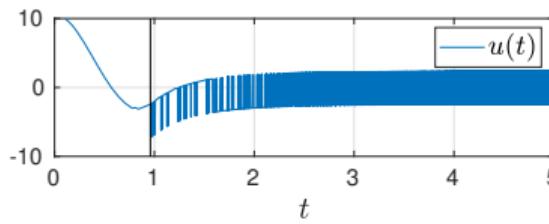
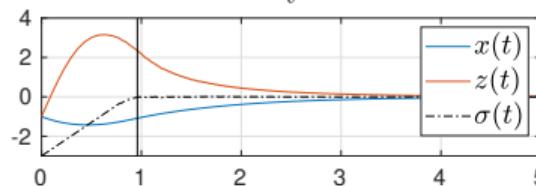
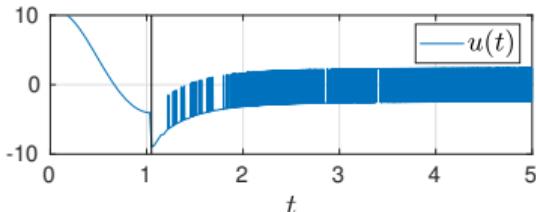
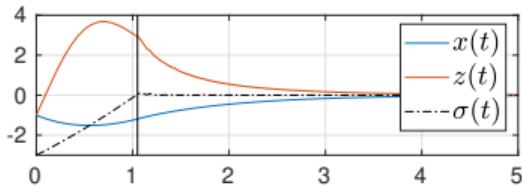
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We observe that

- $\sigma$  converges to zero in finite-time
- Afterwards  $(x, z)$  asymptotically approach the origin
- Since the ordinary differential equation is solved numerically,  $\sigma$  is not exactly zero!



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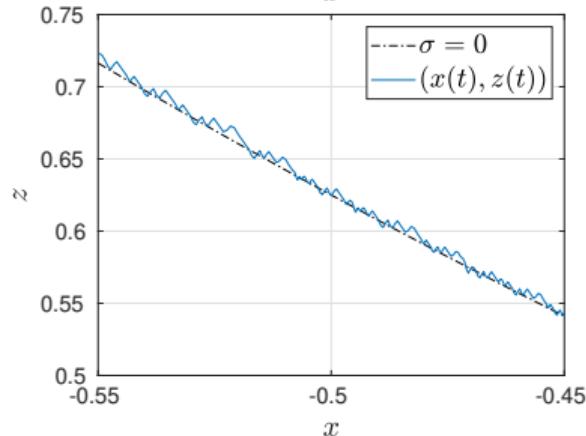
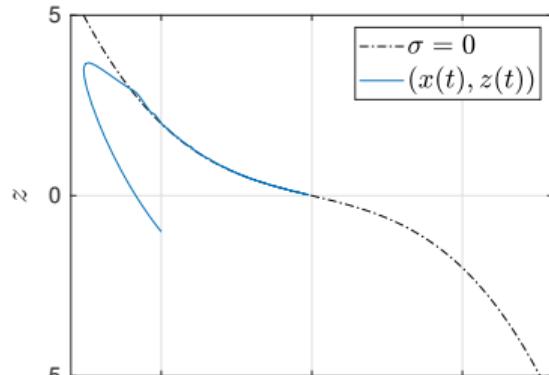
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Convergence structure:

~ Similar to backstepping/forwarding



# Introduction to Nonlinear Control: Stability, control design, and estimation

## Part I: Dynamical Systems

1. Nonlinear Systems - Fundamentals & Examples
2. Nonlinear Systems - Stability Notions
3. Linear Systems and Linearization
4. Frequency Domain Analysis
5. Discrete Time Systems
6. Absolute Stability
7. Input-to-State Stability

## Part II: Controller Design

8. LMI Based Controller and Antiwindup Designs
9. Control Lyapunov Functions
10. Sliding Mode Control
11. Adaptive Control
12. Introduction to Differential Geometric Methods
13. Output Regulation
14. Optimal Control
15. Model Predictive Control

## Part III: Observer Design & Estimation

16. Observer Design for Linear Systems
17. Extended & Unscented Kalman Filter & Moving Horizon Estimation
18. Observer Design for Nonlinear Systems