Introduction to Nonlinear Control

Stability, control design, and estimation

Christopher M. Kellett & Philipp Braun

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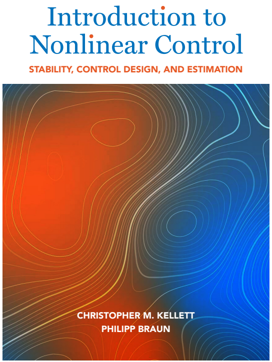

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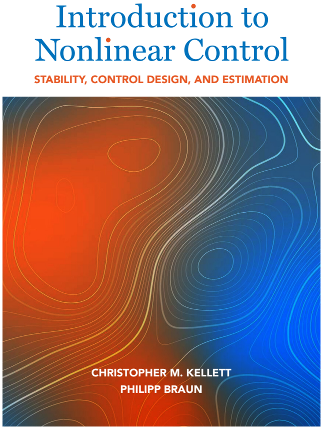
Part I: Dynamical Systems



Control Lyapunov Functions

Recall the dynamical system consider:

$$\dot{x} = f(x) \quad \text{with} \quad f(0) = 0, \quad x \subset \mathbb{R}^n$$

Theorem (Asymptotic stability theorem)

Suppose there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *and* $\rho \in \mathcal{P}$ *such that, for all* $x \in \mathbb{R}^n$*,*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$\langle \nabla V(x), f(x) \rangle \leq -\rho(|x|)$$

Then the origin is (globally) asymptotically stable.

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$$\dot{x} = f(x, u)$$

- Goal: Define $u = k(x)$ asymptotically stabilizing the origin.

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- In terms of a feedback law $u = k(x)$,

$$\frac{d}{dt} V(x(t)) = \langle \nabla V(x), f(x, k(x)) \rangle < 0, \qquad \forall\, x \neq 0$$

$\rightsquigarrow$ V is a Lyapunov function for $\dot{x} = f(x, k(x)) = \tilde{f}(x)$

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Definition (Control Lyapunov function (CLF))

Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called control Lyapunov function for $\dot{x} = f(x, u)$ if

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \qquad \forall\, x \in \mathbb{R}^n,$$

and for all $x \in \mathbb{R}^n \backslash \{0\}$ there exists $u \in \mathbb{R}^m$ such that

$$\langle \nabla V(x), f(x, u) \rangle < 0.$$

Sontag's Universal Formula

Question:

- If we know a CLF, how to define a feedback law?

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Consider

- a control affine system ($u \in \mathbb{R}$)

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with corresponding CLF V, i.e.,

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Alternative representation of the decrease condition:

$$L_f V(x) < 0 \quad \forall \ x \in \mathbb{R}^n \backslash \{0\} \text{ such that } L_g V(x) = 0$$

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Then

- for $\kappa > 0$ we can define the feedback law

$$k(x) = \begin{cases} -\left(\kappa + \frac{L_f V(x) + \sqrt{L_f V(x)^2 + L_g V(x)^4}}{L_g V(x)^2}\right) L_g V(x), & L_g V(x) \neq 0 \\ 0, & L_g V(x) = 0 \end{cases}$$

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The feedback law

- asymptotically stabilizes the origin
- inherits the regularity properties of the CLF except at the origin
- is continuous at the origin if the CLF satisfies a small control property (i.e., $|k(x)| \to 0$ for $|x| \to 0$)

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Note that: Formula known as

- Universal formula
- Sontag's formula

(Derived by Eduardo Sontag)

Backstepping

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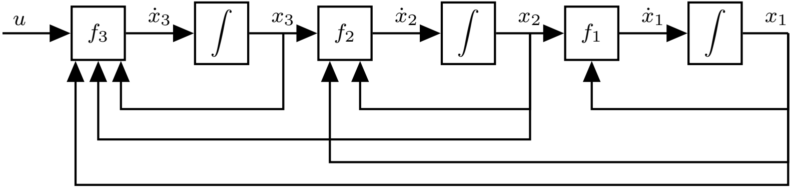
Systems in *strict feedback form*:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2, x_3)\\ &\vdots\\ \dot{x}_{n-1} &= f_{n-1}(x_1, x_2, \ldots, x_{n-1}, x_n)\\ \dot{x}_n &= f_n(x_1, x_2, \ldots, x_n, u).\end{aligned}$$



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$$\dot{x} = x^3 + x\xi, \qquad \dot{\xi} = u.$$

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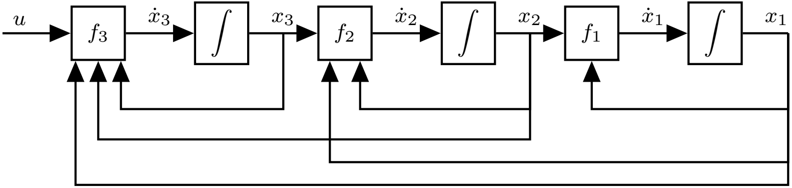

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Backstepping idea:

- Treat ξ as an input to define feedback law $k_\xi(x)$ stabilizing the x-dynamics and to find corresponding CLF $V_1(x)$
- Define error variable $z = \xi - k_\xi(x)$

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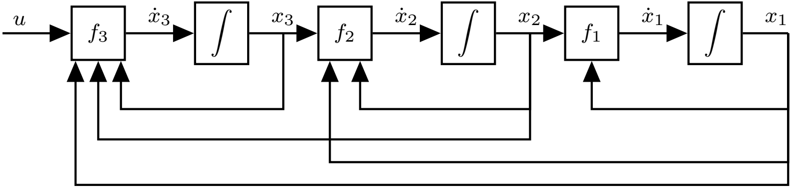

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- Define error variable $z = \xi - k_\xi(x)$
- Derive error dynamics $\dot{z} = \frac{d}{dt}(\xi - k_\xi(x))$
- Stabilize error dynamics through feedback law $k(x, z)$ and define corresponding CLF $V_2(z)$
- The feedback law stabilizes the original (x, ξ)-dynamics and a $V_1(x) + V_2(z)$ is a corresponding CLF.

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