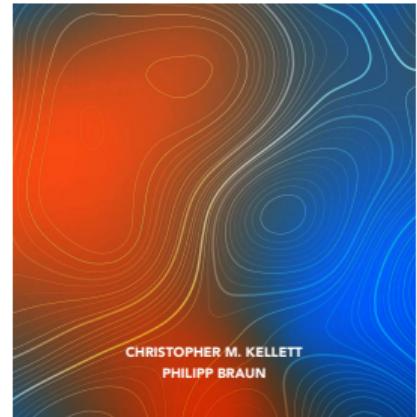


Introduction to Nonlinear Control

Stability, control design, and estimation

Christopher M. Kellett & Philipp Braun

Introduction to
Nonlinear Control
STABILITY, CONTROL DESIGN, AND ESTIMATION



Part I: Dynamical Systems

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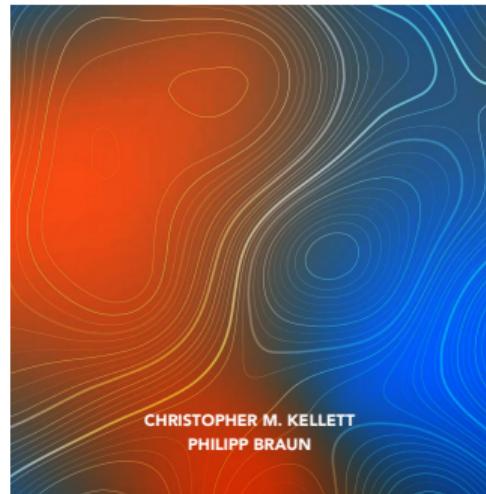
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Introduction to Nonlinear Control

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Control Lyapunov Functions

Recall the dynamical system consider:

$$\dot{x} = f(x) \quad \text{with} \quad f(0) = 0, \quad x \in \mathbb{R}^n$$

Theorem (Asymptotic stability theorem)

Suppose there exists a continuously differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{P}$ such that, for all $x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

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$$\dot{x} = f(x, u)$$

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- In terms of a feedback law $u = k(x)$,

$$\frac{d}{dt} V(x(t)) = \langle \nabla V(x), f(x, k(x)) \rangle < 0, \quad \forall x \neq 0$$

⇒ V is a Lyapunov function for $\dot{x} = f(x, k(x)) = \tilde{f}(x)$

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Definition (Control Lyapunov function (CLF))

Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. A continuously differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is called control Lyapunov function for $\dot{x} = f(x, u)$ if

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad \forall x \in \mathbb{R}^n,$$

and for all $x \in \mathbb{R}^n \setminus \{0\}$ there exists $u \in \mathbb{R}^m$ such that

$$\langle \nabla V(x), f(x, u) \rangle < 0.$$

Sontag's Universal Formula

Question:

- If we know a CLF, how to define a feedback law?

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- a control affine system ($u \in \mathbb{R}$)

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with corresponding CLF V , i.e.,

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Alternative representation of the decrease condition:

$$L_f V(x) < 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\} \text{ such that } L_g V(x) = 0$$

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- for $\kappa > 0$ we can define the feedback law

$$k(x) = \begin{cases} -\left(\kappa + \frac{L_f V(x) + \sqrt{L_f V(x)^2 + L_g V(x)^4}}{L_g V(x)^2}\right) L_g V(x), & L_g V(x) \neq 0 \\ 0, & L_g V(x) = 0 \end{cases}$$

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The feedback law

- asymptotically stabilizes the origin
- inherits the regularity properties of the CLF except at the origin
- is continuous at the origin if the CLF satisfies a small control property (i.e., $|k(x)| \rightarrow 0$ for $|x| \rightarrow 0$)

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Note that: Formula known as

- Universal formula
- Sontag's formula

(Derived by Eduardo Sontag)

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Question:

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Backstepping

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Systems in *strict feedback form*:

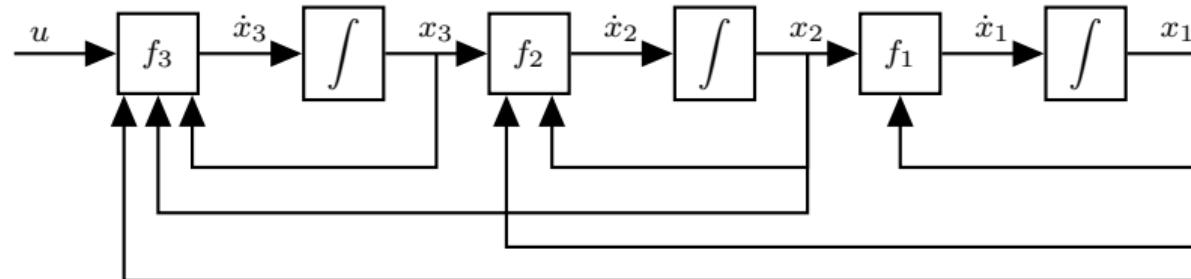
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$$\dot{x}_n = f_n(x_1, x_2, \dots, x_n, u).$$



Example:

$$\dot{x} = x^3 + x\xi, \quad \dot{\xi} = u.$$

Backstepping

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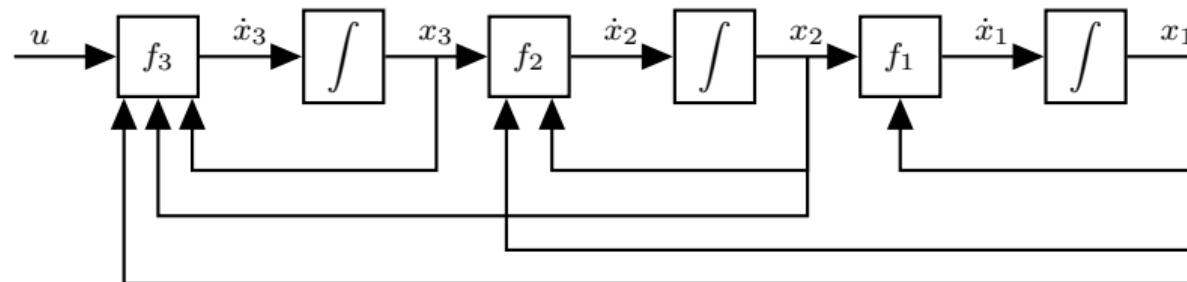
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Backstepping idea:

- Treat ξ as an input to define feedback law $k_\xi(x)$ stabilizing the x -dynamics and to find corresponding CLF $V_1(x)$
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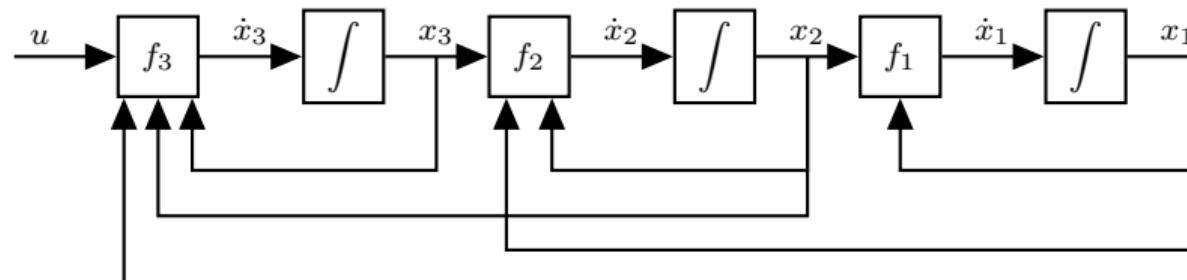
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- Derive error dynamics $\dot{z} = \frac{d}{dt}(\xi - k_\xi(x))$
- Stabilize error dynamics through feedback law $k(x, z)$ and define corresponding CLF $V_2(z)$
- The feedback law stabilizes the original (x, ξ) -dynamics and a $V_1(x) + V_2(z)$ is a corresponding CLF.

Introduction to Nonlinear Control: Stability, control design, and estimation

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4. Frequency Domain Analysis
5. Discrete Time Systems
6. Absolute Stability
7. Input-to-State Stability

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9. Control Lyapunov Functions
10. Sliding Mode Control
11. Adaptive Control
12. Introduction to Differential Geometric Methods
13. Output Regulation
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15. Model Predictive Control

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