

Introduction to Nonlinear Control

Stability, control design, and estimation

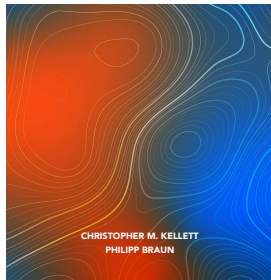
Christopher M. Kellett & Philipp Braun



Introduction to Nonlinear Control

STABILITY, CONTROL DESIGN, AND ESTIMATION

CHRISTOPHER M. KELLETT
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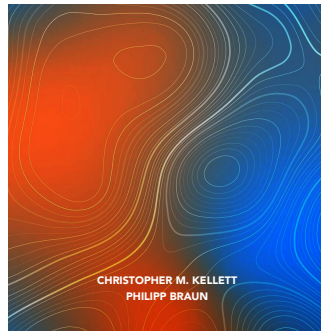
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Introduction to Nonlinear Control

STABILITY, CONTROL DESIGN, AND ESTIMATION



$\mathcal{L}_2$ -Gain Optimization for Linear Systems

Consider: $\dot{x} = Ax$

- Assume that the origin is exponentially stable
- Then, for $Q > 0$ there exists $P > 0$ satisfying

$$A^T P + PA = -Q \quad (\text{Lyapunov equation})$$

- For $V(x) = x^T P x$ (Lyapunov function) it holds that

$$\dot{V}(x) = x^T (A^T P + PA)x = -x^T Q x < 0, \quad x \neq 0.$$

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We note that:

- The inequality of the decrease is important not the equality of the Lyapunov equation

↪ For given A , consider the linear matrix inequality (LMI)

$$0 < P$$

$$A^T P + PA < 0$$

- Advantage: Q is a degree of freedom
- “Optimal” Q and P can be obtained

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$$\begin{array}{ll} \min_{P, k} & k \\ \text{subject to} & 0 < k \\ & 0 < P - \alpha I \\ & 0 > P - (k + \alpha)I \\ & 0 > A^T P + PA. \end{array}$$

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Here:

- $\alpha > 0$ to ensure that P is not arbitrarily small
- Third constraint to ensure that P is not arbitrarily large

Toolboxes in Matlab:

- CVX, SOSTOOLS, YALMIP

Approximation: ($\varepsilon > 0$)

$$\min_{P, k} \quad k$$

$$\begin{aligned} \text{subject to} \quad & 0 \leq k \\ & 0 \leq P - \alpha I - \varepsilon I \\ & 0 \geq P - (k + \alpha)I + \varepsilon I \\ & 0 \geq A^T P + PA + \varepsilon I \end{aligned}$$

Asymptotic Stability and $\mathcal{L}_2$ -Gain Optimization

Consider:

$$\dot{x} = Ax + Ew$$

$$z = Cx + Fw.$$

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- ⇒ If A is Hurwitz then the origin is 0-GAS

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Recall: For A Hurwitz, $Q = -2I$, $P > 0$ solution of the Lyap. equation, $V(x) = x^T Px$, we can derive

$$\dot{V}(x) \leq -x^T x + \gamma^2 w^T w, \quad \gamma = \|PE\|$$

Rearranging terms and integrating (with $x(0) = 0$) yields

$$\begin{aligned} \|x\|_{\mathcal{L}_2[0,t]}^2 &\leq \int_0^t x(\tau)^T x(\tau) d\tau + V(x(t)) \\ &\leq \gamma^2 \int_0^t w(\tau)^T w(\tau) d\tau = \gamma^2 \|w\|_{\mathcal{L}_2[0,t]}^2. \end{aligned}$$

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Slight modification: Suppose we can find $P > 0$, so that

$$\begin{aligned}\dot{V}(x) &= x^T (A^T P + PA)x + 2x^T P E w \\ &< -\gamma \left(\frac{1}{\gamma^2} z^T z - w^T w \right), \quad \forall (x, w) \neq 0\end{aligned}$$

Then we can show that this guarantees

- 0-GAS (since $\dot{V}(x) < 0 \quad \forall x \neq 0$)
- an $\mathcal{L}_2$ -gain bound of $\gamma > 0$ from w to output z ; i.e.,

$$\|z\|_{\mathcal{L}_2[0,t]} \leq \gamma \|w\|_{\mathcal{L}_2[0,t]}$$

The bound again follows by integrating (and $x(0) = 0$):

$$\frac{1}{\gamma} \int_0^t z^T(\tau) z(\tau) d\tau + V(x(t)) \leq \gamma \int_0^t w^T(\tau) w(\tau) d\tau$$

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↪ Can we compute $P > 0$ and $\gamma > 0$ by solving an LMI?

Starting point:

$$x^T (A^T P + PA)x + 2x^T P E w + \frac{1}{\gamma} z^T z - \gamma w^T w < 0$$

Asymptotic Stability and $\mathcal{L}_2$ -Gain Optimization

Starting point:

$$x^T(A^T P + PA)x + 2x^T PEw + \frac{1}{\gamma}z^T z - \gamma w^T w < 0$$

$$x^T(A^T P + PA + \frac{1}{\gamma}C^T C)x + 2x^T(PE + \frac{1}{\gamma}C^T F)w + \gamma w^T F^T Fw < 0$$

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \left(\begin{bmatrix} A^T P + PA & PE \\ E^T P & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ F^T \end{bmatrix} \begin{bmatrix} C & F \end{bmatrix} \right) \begin{bmatrix} x \\ w \end{bmatrix} < 0$$

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In terms of definite matrices ($0 < P$ and):

$$\begin{bmatrix} A^T P + PA & PE \\ E^T P & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ F^T \end{bmatrix} \begin{bmatrix} C & F \end{bmatrix} < 0 \quad (1)$$

Note that:

- For $\gamma > 0$, fixed we know how to solve the LMI to obtain P
- However, we would like to minimize $\gamma > 0$
- The inequality is not linear in γ

Asymptotic Stability and $\mathcal{L}_2$ -Gain Optimization

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Lemma (Schur Complement)

Let $Q \in S^r$ and $R \in S^q$ for $r, q \in \mathbb{N}$ and let $S \in \mathbb{R}^{r \times q}$. Then the matrix condition

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} < 0$$

is equivalent to the matrix conditions

$$R < 0$$

$$Q - SR^{-1}S^T < 0.$$

Asymptotic Stability and $\mathcal{L}_2$ -Gain Optimization

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Here, take $R = -\gamma$, $S = \begin{bmatrix} C & F \end{bmatrix}$ and Q as the leftmost matrix.

Then, (1) is equivalent to

$$\left[\begin{array}{cc|c} A^T P + PA & PE & C^T \\ E^T P & -\gamma I & F^T \\ \hline C & F & -\gamma I \end{array} \right] < 0$$

Note that:

- For $\gamma > 0$, fixed we know how to solve the LMI to obtain P
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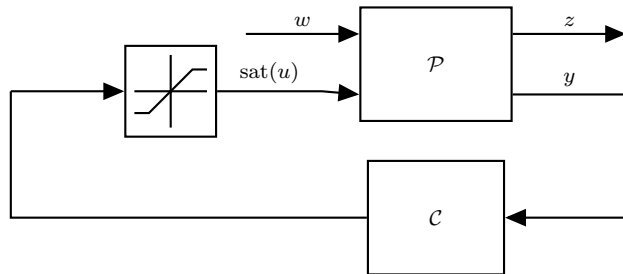
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$$\begin{aligned} R &< 0 \\ Q - SR^{-1}S^T &< 0. \end{aligned}$$

LMI Based Controller and Antiwindup Designs



Plant & Controller:

$$\mathcal{P} : \begin{cases} \dot{x}_p &= A_p x_p + B_p \text{sat}(u) + B_w w \\ y &= C_{p,y} x_p + D_{p,y} w \\ z &= C_{p,z} x_p + D_{p,z} w \end{cases}$$

$$\mathcal{C} : \begin{cases} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c + D_{c,y} y \end{cases}$$

Compact representation: $(x = [x_p^T, x_c^T]^T \in \mathbb{R}^n)$

$$\left[\begin{array}{c|c|c} A & B & E \\ \hline C & D & F \\ \hline K & L & G \end{array} \right] = \left[\begin{array}{cc|c|c} A_p + B_p D_{c,y} C_{p,y} & B_p C_c & -B_p & B_p D_{c,y} D_{p,y} + B_w \\ B_c C_{p,y} & A_c & 0 & B_c D_{p,y} \\ \hline C_{p,z} & 0 & 0 & D_{p,z} \\ \hline D_{c,y} C_{p,y} & C_c & 0 & D_{c,y} D_{p,y} \end{array} \right]$$

$$\begin{aligned} \dot{x} &= Ax + Bq + Ew \\ z &= Cx + Dq + Fw \\ u &= Kx + Lq + Gw \\ q &= u - \text{sat}(u) \end{aligned}$$

Introduction to Nonlinear Control: Stability, control design, and estimation

Part I: Dynamical Systems

1. Nonlinear Systems - Fundamentals & Examples
2. Nonlinear Systems - Stability Notions
3. Linear Systems and Linearization
4. Frequency Domain Analysis
5. Discrete Time Systems
6. Absolute Stability
7. Input-to-State Stability

Part II: Controller Design

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9. Control Lyapunov Functions
10. Sliding Mode Control
11. Adaptive Control
12. Introduction to Differential Geometric Methods
13. Output Regulation
14. Optimal Control
15. Model Predictive Control

Part III: Observer Design & Estimation

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17. Extended & Unscented Kalman Filter & Moving Horizon Estimation
18. Observer Design for Nonlinear Systems