A Run Through Nonlinear Control Topics

Stability, control design, and estimation

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Introduction to Nonlinear Control: Stability, control design, and estimation

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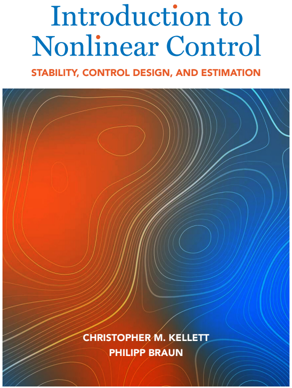

1. Nonlinear Systems – Fundamentals (Dynamical Systems)

(Autonomous) First order differential equations:

$$\dot{x}(t) = \tfrac{d}{dt}x(t) = f(x(t)), \quad f : \mathbb{R}^n \to \mathbb{R}^n \qquad (1)$$

- A *solution* is an absolutely continuous function that satisfies (1) for almost all t.

Non-autonomous/time-varying system:

$$\dot{x}(t) = f(t, x(t)), \qquad f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$$

Systems with external inputs $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$:

$$\dot{x} = f(x, u), \qquad \dot{x} = f(x, w),$$

- $u : \mathbb{R}^n \to \mathbb{R}^m, \quad x \mapsto u(x)$ $\leftarrow$ degree of freedom
- $w : \mathbb{R} \to \mathbb{R}^m, \quad t \mapsto w(t)$ $\leftarrow$ exogenous signal (disturbance or reference)

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- $w : \mathbb{R} \to \mathbb{R}^m, \quad t \mapsto w(t)$ ← exogenous signal (disturbance or reference)

Definition (Equilibrium, $\dot{x} = 0$)

The point $x^e \in \mathbb{R}^n$ is called equilibrium of the system $\dot{x} = f(x)$ or $\dot{x} = f(t, x)$, respectively, if

$$\tfrac{d}{dt}x(t) = f(x^e) = 0,$$
$$\tfrac{d}{dt}x(t) = f(t, x^e) = 0 \qquad \forall t \in \mathbb{R}_{\geq 0}.$$

The pair $(x^e, u^e) \in \mathbb{R}^n \times \mathbb{R}^m$ is called an equilibrium pair of the system $\dot{x} = f(x, u)$ if

$$\tfrac{d}{dt}x(t) = f(x^e, u^e) = 0.$$

- Without loss of generality $x^e = 0$ (or $(x^e, u^e) = 0$).
- Achieved through coordinate transf. $z = x - x^e$, i.e.,

$$\hat{f}(z) \doteq f(z + x^e) \qquad \text{yields} \qquad \dot{z} = \hat{f}(z)$$

where ($z^e = 0$)

$$\hat{f}(z^e) = f(z^e + x^e) = f(x^e) = 0$$

1. Nonlinear Systems – Fundamentals (Comparison Functions)

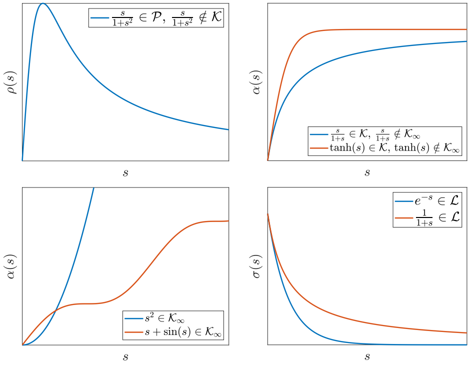
Definition (Class-$\mathcal{P}, \mathcal{K}, \mathcal{K}_\infty, \mathcal{L}, \mathcal{KL}$ functions)

- A continuous function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be positive definite ($\rho \in \mathcal{P}$) if $\rho(0) = 0$ and $\rho(s) > 0 \; \forall \; s \in \mathbb{R}_{>0}$.
- $\alpha \in \mathcal{P}$ is said to be of class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if α strictly increasing.
- $\alpha \in \mathcal{K}$ is said to be of class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if $\lim_{s\to\infty} \alpha(s) = \infty$.
- A continuous function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class-$\mathcal{L}$ ($\sigma \in \mathcal{L}$) if σ is strictly decreasing and $\lim_{s\to\infty} \sigma(s) = 0$.
- A continuous function $\beta : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class-$\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if for each fixed $t \in \mathbb{R}_{\geq 0}$, $\beta(\cdot, t) \in \mathcal{K}_\infty$ and for each fixed $s \in \mathbb{R}_{>0}$, $\beta(s, \cdot) \in \mathcal{L}$.

$$\rightsquigarrow \mathcal{K}_\infty \subset \mathcal{K} \subset \mathcal{P}$$

Some properties:

- Class-$\mathcal{K}_\infty$ functions are invertible.
- If $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ then
$$\alpha(s) \doteq \alpha_1(\alpha_2(s)) = \alpha_1 \circ \alpha_2(s) \in \mathcal{K}_\infty.$$
- If $\alpha \in \mathcal{K}$, $\sigma \in \mathcal{L}$ then $\alpha \circ \sigma \in \mathcal{L}$.



2. Nonlinear Systems – Stability Notions (Definitions)

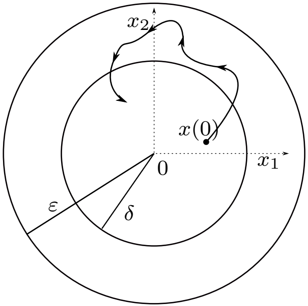
Consider

$$\dot{x} = f(x), \qquad \text{(with } f(0) = 0\text{)}$$

Definition (Stability)

The origin is *(Lyapunov) stable* if, for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $|x(0)| \leq \delta$ then, for all $t \geq 0$,

$$|x(t)| \leq \varepsilon.$$



Equivalent Definition:

The origin is stable if there exists $\alpha \in \mathcal{K}$ and an open neighborhood around the origin $\mathcal{D} \subset \mathbb{R}^n$, such that

$$|x(t)| \leq \alpha(|x(0)|), \qquad \forall\, t \geq 0, \;\; \forall\, x_0 \in \mathcal{D}.$$

Definition (Instability)

The origin is *unstable* if it is not stable.

2. Nonlinear Systems – Stability Notions (Definitions)

Consider $\dot{x} = f(x)$ with $f(0) = 0$

Definition (Attractivity)

The origin is *attractive* if there exists $\delta > 0$ such that if $|x(0)| < \delta$ then

$$\lim_{t \to \infty} x(t) = 0.$$

Definition (Asymptotic stability)

The origin is *asymptotically stable* if it is both stable and attractive.

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Definition (Asymptotic stability)

The origin is *asymptotically stable* if it is both stable and attractive.

Definition ($\mathcal{KL}$-stability)

The system is said to be $\mathcal{KL}$*-stable* if there exists $\delta > 0$ and $\beta \in \mathcal{KL}$ such that if $|x(0)| \leq \delta$ then for all $t \geq 0$,

$$|x(t)| \leq \beta(|x(0)|, t).$$

Proposition

The origin is asymptotically stable if and only if it is $\mathcal{KL}$-stable.

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The system is said to be *$\mathcal{KL}$-stable* if there exists $\delta > 0$ and $\beta \in \mathcal{KL}$ such that if $|x(0)| \leq \delta$ then for all $t \geq 0$,

$$|x(t)| \leq \beta(|x(0)|, t).$$

Proposition

The origin is asymptotically stable if and only if it is $\mathcal{KL}$-stable.

Definition (Exponential stability)

The origin is *exponentially stable* for $\dot{x} = f(x)$ if there exist $\delta, \lambda, M > 0$ such that if $|x(0)| \leq \delta$ then for all $t \geq 0$,

$$|x(t)| \leq M|x(0)|e^{-\lambda t}. \quad (2)$$

Example: The origin of

- $\dot{x} = x$ is unstable
- $\dot{x} = 0$ is stable
- $\dot{x} = -x^3$ is asymptotically stable
- $\dot{x} = -x$ is exponentially stable

2. Nonlinear Systems – Stability Notions (Lyapunov's Second Method)

Consider $\dot{x} = f(x)$ with $f(0) = 0$

Theorem (Lyapunov stability theorem)

Let $V : \mathbb{R}^n \to \mathbb{R}$, cont. differentiable and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that, for all $x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \qquad \textit{and} \qquad \langle \nabla V(x), f(x) \rangle \leq 0.$$

Then the origin is globally stable.

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Theorem (Asymptotic stability theorem)

Let $V : \mathbb{R}^n \to \mathbb{R}$, cont. differentiable, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and $\rho \in \mathcal{P}$ such that, for all $x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad \text{and} \quad \langle \nabla V(x), f(x) \rangle \leq -\rho(|x|).$$

Then the origin is globally asymptotically stable.

Theorem (Exponential stability theorem)

Let $V : \mathbb{R}^n \to \mathbb{R}$, cont. differentiable, constants $\lambda_1, \lambda_2, c > 0$ and $p \geq 1$ such that, for all $x \in \mathbb{R}^n$

$$\lambda_1 |x|^p \leq V(x) \leq \lambda_2 |x|^p \quad \text{and} \quad \langle \nabla V(x), f(x) \rangle \leq -cV(x).$$

Then the origin is globally exponentially stable.

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Then the origin is globally exponentially stable.

Theorem (Partial Convergence)

Let $V : \mathbb{R}^n \to \mathbb{R}$, cont. differentiable, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that, for all $x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad \text{and} \quad \langle \nabla V(x), f(x)\rangle \leq -W(x).$$

Then $\lim_{t\to\infty} W(x(t)) = 0.$

Theorem (Lyapunov theorem for instability)

Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ cont. differentiable and $\varepsilon > 0$ such that

$$\langle \nabla V(x), f(x)\rangle > 0 \quad \forall x \in \mathcal{B}_\varepsilon \backslash \{0\}$$

Then the origin is (completely) unstable.

Theorem (Chetaev's theorem)

Let $V : \mathbb{R}^n \to \mathbb{R}$ be cont. differentiable with $V(0) = 0$ and $\mathcal{O}_r = \{x \in \mathcal{B}_r(0) |\ V(x) > 0\} \neq \emptyset$ for all $r > 0$. If for certain $r > 0$,

$$\langle \nabla V(x), f(x)\rangle > 0, \qquad \forall\, x \in \mathcal{O}_r$$

then the origin is unstable.

2. Nonlinear Systems – Stability Notions (Lypunov's Second Method)

Intuition:

- Lyapunov functions represent energy associated with the state of a system
- If energy is (strictly) decreasing, then an equilibrium is (symptotically) stable

$$\dot{V}(x(t)) = \langle \nabla V(x), f(x) \rangle < 0 \qquad \forall x \neq 0$$

Extensions:

- (LaSalle's) Invariance principles
- Similar results for time-varying systems
- Converse Lyapunov results (i.e., asymptotic stability implies existence of Lyapunov function)

3. Linear Systems (Stability)

Linear Systems:

$$\dot{x} = Ax, \qquad A \in \mathbb{R}^{n \times n}$$

Theorem

For the linear system $\dot{x} = Ax$, the following are equivalent:

1. *The origin is asymptotically/exponentially stable;*
2. *All eigenvalues of A have strictly negative real parts;*
3. *For every $Q > 0$, there exists a unique $P > 0$, satisfying the Lyapunov equation*

$$A^T P + PA = -Q.$$

Lyapunov Function:

$$V(x) = x^T P x$$

It holds that:

$$\dot{V}(x(t)) = \tfrac{d}{dt}\left(x^T P x\right) = \dot{x}^T P x + x^T P \dot{x}$$
$$= x^T A^T P x + x^T P A x = -x^T Q x$$

Consider:

$$\dot{x} = f(x), \qquad f(0) = 0, \quad f \text{ cont. differentiable}$$

Define (Jacobian evaluated at the origin):

$$A = \left[\frac{\partial f(x)}{\partial x}\right]_{x=0}$$

Linearization of $\dot{x} = f(x)$ at $x = 0$:

$$\dot{z}(t) = Az(t)$$

Theorem

Consider $\dot{x} = f(x)$ (f cont. differentiable) and its linearization $\dot{z} = Az$. If the origin $z^e = 0$ of $\dot{z} = Az$ is globally exponentially stable then the origin $x^e = 0$ of $\dot{x} = f(x)$ is locally exponentially stable.

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Semidefinite programming:

$$\begin{array}{c} \varepsilon I \le P \\ A^T P + PA \le -\varepsilon I \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \varepsilon |x|^2 \le V(x) \\ \langle \nabla V(x), Ax \rangle \le -\varepsilon |x|^2 \end{array}$$

$\rightsquigarrow$ Construction can be extended to systems with polynomial right-hand side

5. Discrete Time Systems (Fundamentals)

Discrete time systems:

$$\begin{aligned} x_d(k+1) &= F(x_d(k), u_d(k)), \quad x_d(0) = x_{d,0} \in \mathbb{R}^n \\ y_d(k) &= H(x_d(k), u_d(k)) \end{aligned}$$

Time-varying discrete time system ($k \geq k_0 \geq 0$):

$$x_d(k+1) = F(k, x_d(k)), \quad x_d(k_0) = x_{d,0} \in \mathbb{R}^n$$

Time invariant discrete time systems without input:

$$x_d(k+1) = F(x_d(k)), \quad x_d(0) = x_{d,0} \in \mathbb{R}^n,$$

Shorthand notation for difference equations:

$$x_d^+ = F(x_d, u_d),$$

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Shorthand notation for difference equations:

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Definition (Equilibrium)

- The point $x_d^e \in \mathbb{R}^n$ is called equilibrium if $x_d^e = F(x_d^e)$ or $x_d^e = F(k, x_d^e)$ for all $k \in \mathbb{N}$ is satisfied.
- The pair $(x_d^e, u_d^e) \in \mathbb{R}^n \times \mathbb{R}^m$ is called equilibrium pair of the system if $x_d^e = F(x_d^e, u_d^e)$ holds.

Again, without loss of generality we can shift the equilibrium (pair) to the origin.

Definition (Equilibrium, $\dot{x} = 0$)

The point $x^e \in \mathbb{R}^n$ is called an equilibrium of the system $\dot{x} = f(x)$ if $\frac{d}{dt}x(t) = f(x^e) = 0$

5. Discrete Time Systems (Stability)

Discrete time systems: Consider

$$x^+ = F(x), \qquad x(0) = x_0 \in \mathbb{R}^n$$

Definition ($\mathcal{KL}$-stability)

The origin of the discrete time system is is globally asymptotically stable, or alternatively $\mathcal{KL}$-stable, if there exists $\beta \in \mathcal{KL}$ such that

$$|x(k)| \leq \beta(|x(0)|, k), \qquad \forall\, k \in \mathbb{N},$$

is satisfied for all $x(0) \in \mathbb{R}^n$.

Theorem (Lyapunov stability theorem)

Suppose there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that, for all $x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$V(F(x)) - V(x) \leq 0$$

Then the origin is stable.

Continuous time systems: Consider

$$\dot{x} = f(x), \qquad x(0) = x_0 \in \mathbb{R}^n$$

Definition ($\mathcal{KL}$-stability)

The origin of the continuous time system is globally asymptotically stable, or alternatively $\mathcal{KL}$-stable, if there exists $\beta \in \mathcal{KL}$ such that

$$|x(t)| \leq \beta(|x(0)|, t), \qquad \forall\, t \in \mathbb{R}_{\geq 0},$$

is satisfied for all $x(0) \in \mathbb{R}^n$.

Theorem (Lyapunov stability theorem)

Suppose there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that, for all $x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$\langle \nabla V(x), f(x) \rangle \leq 0$$

Then the origin is stable.

5. Discrete Time Systems (Linear systems)

Consider the discrete time linear system

$$x^+ = Ax, \qquad x(0) \in \mathbb{R}^n \qquad \text{[Solution } x(k) = A^k x(0)\text{]}$$

Theorem

The following properties are equivalent:

1. *The origin $x^e = 0$ is exponentially stable;*
2. *The eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ of A satisfy $|\lambda_i| < 1$ for all $i = 1, \dots, n$; and*
3. *For $Q > 0$ there exists a unique $P > 0$ satisfying the discrete time Lyapunov equation*

$$A^T P A - P = -Q.$$

A matrix A which satisfies $|\lambda_i| < 1$ for all $i = 1, \dots, n$ is called a *Schur matrix*.

Consider the continuous time linear system

$$\dot{x} = Ax, \qquad x(0) \in \mathbb{R}^n \qquad \text{[Solution } x(t) = e^{At} x(0)\text{]}$$

Theorem

The following properties are equivalent:

1. *The origin $x^e = 0$ is exponentially stable;*
2. *The eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ of A satisfy $\lambda_i \in \mathbb{C}^-$ for all $i = 1, \dots, n$; and*
3. *For $Q > 0$ there exists a unique $P > 0$ satisfying the continuous time Lyapunov equation*

$$A^T P + P A = -Q.$$

A matrix A which satisfies $\lambda_i \in \mathbb{C}^-$ for all $i = 1, \dots, n$ is called a *Hurwitz matrix*.

5. Discrete Time Systems (Sampling)

Derivative for continuously differentiable function:

$$\frac{d}{dt}x(t) = \lim_{\Delta \to 0} \frac{x(t+\Delta) - x(t)}{\Delta}$$

Difference quotient (for $\Delta > 0$ small):

$$\frac{x(t+\Delta) - x(t)}{\Delta} \approx \frac{d}{dt}x(t) = \dot{x}(t) = f(x(t), u(t))$$

or equivalently

$$x(t+\Delta) \approx x(t) + \Delta f(x(t), u(t))$$

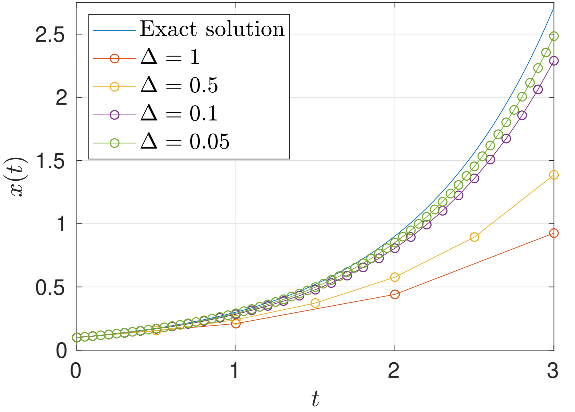
Approximated discrete time system (identify t with $k \cdot \Delta$)

$$x_d^+ = F(x_d, u_d) \doteq x_d + \Delta f(x_d, u_d)$$

⇝ This discretization is known as (explicit) *Euler method*.

Approximation of $\dot{x} = 1.1x$

Euler discretization: $x^+ = (1 + \Delta 1.1)x$



5. Discrete Time Systems (Runge-Kutta Methods)

- Consider
$$\dot{x} = g(t, x).$$
- Runge-Kutta update formula:
$$x(t+\Delta) = x(t) + \Delta \sum_{i=1}^{s} b_i k_i$$
where

$$k_1 = g(t, x(t))$$
$$k_2 = g(t + c_2\Delta, x + \Delta(a_{21}k_1))$$
$$k_3 = g(t + c_3\Delta, x + \Delta(a_{31}k_1 + a_{32}k_2))$$
$$\vdots$$
$$k_s = g(t + c_s\Delta, x + \Delta(a_{s1}k_1 + a_{s2}k_2 + \cdots + a_{s(s-1)}k(s)))$$

- $s \in \mathbb{N}$ (stage); $a_{ij}, b_\ell, c_i \in \mathbb{R}$, $1 \leq j < i \leq s$, $1 \leq \ell \leq s$ (given parameters)
- The case $f(x, u)$ for sample-and-hold inputs $u(t+\delta) = u_d \in \mathbb{R}^m$ for all $\delta \in [0, \Delta)$ is covered through
$$g(t, x(t)) = f(x(t), u_d)$$

5. Discrete Time Systems (Runge-Kutta Methods)

- Consider

$$\dot{x} = g(t, x).$$

- Runge-Kutta update formula:

$$x(t+\Delta) = x(t) + \Delta \sum_{i=1}^{s} b_i k_i$$

where

$$\begin{aligned} k_1 &= g(t, x(t)) \\ k_2 &= g(t + c_2\Delta, x + \Delta(a_{21}k_1)) \\ k_3 &= g(t + c_3\Delta, x + \Delta(a_{31}k_1 + a_{32}k_2)) \\ &\vdots \\ k_s &= g(t + c_s\Delta, x + \Delta(a_{s1}k_1 + a_{s2}k_2 + \cdots + a_{s(s-1)}k(s))) \end{aligned}$$

- $s \in \mathbb{N}$ (stage); $a_{ij}, b_\ell, c_i \in \mathbb{R}$, $1 \leq j < i \leq s$, $1 \leq \ell \leq s$ (given parameters)
- The case $f(x, u)$ for sample-and-hold inputs $u(t+\delta) = u_d \in \mathbb{R}^m$ for all $\delta \in [0, \Delta)$ is covered through

$$g(t, x(t)) = f(x(t), u_d)$$

- Butcher tableau:

$$\begin{array}{c|ccccc} 0 & & & & & \\ c_2 & a_{21} & & & & \\ c_3 & a_{31} & a_{32} & & & \\ \vdots & \vdots & & \ddots & & \\ c_s & a_{s1} & a_{s2} & \cdots & a_{s(s-1)} & \\ \hline & b_1 & b_2 & \cdots & b_{s-1} & b_s \end{array}$$

$\rightsquigarrow$ c_i is only necessary for time-varying systems

- Examples: The Euler and the Heun method

$$\begin{array}{c|c} 0 & \\ \hline & 1 \end{array} \quad \text{and} \quad \begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

- Heun Method: Update of x in three steps

$$\begin{aligned} k_1 &= f(x(t), u_d), \\ k_2 &= f(x(t) + \Delta k_1, u_d), \\ x(t+\Delta) &= x(t) + \Delta\left(\tfrac{1}{2}k_1 + \tfrac{1}{2}k_2\right). \end{aligned}$$

The function `ode23.m` relies on the Butcher tableaus

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ \frac{3}{4} & 0 & \frac{3}{4} & \\ \hline & \frac{2}{9} & \frac{1}{3} & \frac{4}{9} \end{array} \quad \text{and} \quad \begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{3}{4} & 0 & \frac{3}{4} & & \\ 1 & \frac{2}{9} & \frac{1}{3} & \frac{4}{9} & \\ \hline & \frac{7}{24} & \frac{1}{4} & \frac{1}{3} & \frac{1}{8} \end{array}$$

- One scheme is used to approximate $x(t+\Delta)$.
- The second scheme is needed to approximate the error, to select the step size Δ.

The function `ode45.m` relies on the Butcher tableaus

$$\begin{array}{c|ccccccc} 0 & & & & & & & \\ \frac{1}{5} & \frac{1}{5} & & & & & & \\ \frac{3}{10} & \frac{3}{40} & \frac{9}{40} & & & & & \\ \frac{4}{5} & \frac{44}{45} & -\frac{56}{15} & \frac{32}{9} & & & & \\ \frac{8}{9} & \frac{19372}{6561} & -\frac{25360}{2187} & \frac{64448}{6561} & -\frac{212}{729} & & & \\ 1 & \frac{9017}{3168} & -\frac{355}{33} & \frac{46732}{5247} & \frac{49}{176} & -\frac{5103}{18656} & & \\ 1 & \frac{35}{384} & 0 & \frac{500}{1113} & \frac{125}{192} & -\frac{2187}{6784} & \frac{11}{84} & \\ \hline & \frac{35}{384} & 0 & \frac{500}{1113} & \frac{125}{192} & -\frac{2187}{6784} & \frac{11}{84} & 0 \end{array}$$

$$\begin{array}{c|ccccccc} \hline & \frac{5179}{57600} & 0 & \frac{7571}{16695} & \frac{393}{640} & -\frac{92097}{339200} & \frac{187}{2100} & \frac{1}{40} \end{array}$$

7. Input-to-State stability (Definition & Motivation)

Input-to-state stability (ISS) for nonlinear systems:

$$\dot{x} = f(x, w), \quad x(0) = x_0 \in \mathbb{R}^n$$
$$w \in \mathcal{W} = \{w : \mathbb{R}_{\geq 0} \to \mathbb{R}^m |\ w \text{ essentially bounded}\}.$$

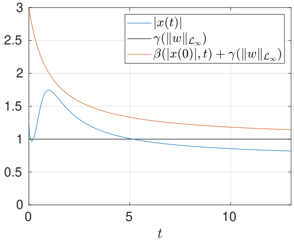
Definition (Input-to-state stability)

The system is said to be *input-to-state stable (ISS)* if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that solutions satisfy

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma\left(\|w\|_{\mathcal{L}_\infty}\right)$$

for all $x \in \mathbb{R}^n$, $w \in \mathcal{W}$, and $t \geq 0$.

- $\gamma \in \mathcal{K}$: *ISS-gain*;
- $\beta \in \mathcal{KL}$: *transient bound*.



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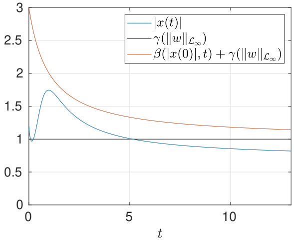

Example

Consider the nonlinear/bilinear system:

$$\dot{x} = -x + xw.$$

- The system is 0-input globally asymptotically stable (since $w = 0$ implies $\dot{x} = -x$ and so $x(t) = x(0)e^{-t}$)
- However, consider the bounded input/disturbance $w = 2$. Then $\dot{x} = x$ and so $x(t) = x(0)e^{t}$.
- Consequently, it is impossible to find $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$|x(t)| = |x(0)|e^{t} \leq \beta(|x(0)|, t) + \gamma(2).$$

7. Input-to-State Stability (Lyapunov Characterizations)

Definition (Input-to-state stability)

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for all $x \in \mathbb{R}^n$, $w \in \mathcal{W}$, and $t \geq 0$.

Theorem (ISS-Lyapunov function)

$\dot{x} = f(x, w)$ is ISS if and only if there exist a cont. differentiable fcn. $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2, \alpha_3, \sigma \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{R}^n$ and all $w \in \mathbb{R}^m$

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$\langle \nabla V(x), f(x, w) \rangle \leq -\alpha_3(|x|) + \sigma(|w|)$$

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$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$\langle \nabla V(x), f(x,w)\rangle \leq -\alpha_3(|x|) + \sigma(|w|)$$

Example

Consider

$$\dot{x} = f(x,w) = -x - x^3 + xw, \quad x(0) = x_0 \in \mathbb{R}$$

The candidate ISS-Lyapunov function $V(x) = \frac{1}{2}x^2$:

$$\begin{aligned} \langle \nabla V(x), f(x,w)\rangle &= \langle x, -x - x^3 + xw\rangle \\ &= -x^2 - x^4 + x^2 w \\ &\leq -x^2 - x^4 + \tfrac{1}{2}x^4 + \tfrac{1}{2}w^2 \\ &= -x^2 - \tfrac{1}{2}x^4 + \tfrac{1}{2}w^2 \end{aligned}$$

- The inequality follows from Young's inequality:

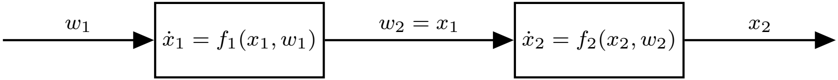
$$yz \leq \frac{1}{2}y^2 + \frac{1}{2}z^2$$

- Define $\alpha(s) \doteq s^2 + \frac{1}{2}s^4$ and $\sigma(s) \doteq \frac{1}{2}s^2$, Then

$$\dot{V}(x) \leq -\alpha(|x|) + \sigma(|w|)$$

i.e., V is an ISS-Lyapunov function, the system is ISS.

7. Input-to-State Stability (Cascade Interconnections)

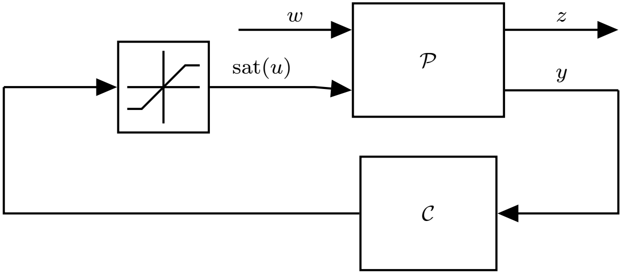


$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, w_1) \\ f_2(x_2, x_1) \end{bmatrix}$$

Theorem (ISS Cascade)

Consider the system with $[x_1, x_2]^T \in \mathbb{R}^n$, $w_2 = x_1$. If each of the subsystems are ISS, then the cascade interconnection is ISS with w_1 as input and x as state.

8. LMI Based Controller and Antiwindup Designs



Plant & Controller:

$$\mathcal{P}: \begin{cases} \dot{x}_p &= A_p x_p + B_p \,\mathrm{sat}(u) + B_w w \\ y &= C_{p,y} x_p + D_{p,y} w \\ z &= C_{p,z} x_p + D_{p,z} w \end{cases}$$

$$\mathcal{C}: \begin{cases} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c + D_{c,y} y \end{cases}$$

Compact representation: ($x = [x_p^T, x_c^T]^T \in \mathbb{R}^n$)

$$\left[\begin{array}{c|c|c} A & B & E \\ \hline C & D & F \\ \hline K & L & G \end{array}\right] = \left[\begin{array}{cc|c|c} A_p + B_p D_{c,y} C_{p,y} & B_p C_c & -B_p & B_p D_{c,y} D_{p,y} + B_w \\ B_c C_{p,y} & A_c & 0 & B_c D_{p,y} \\ \hline C_{p,z} & 0 & 0 & D_{p,z} \\ \hline D_{c,y} C_{p,y} & C_c & 0 & D_{c,y} D_{p,y} \end{array}\right]$$

$$\begin{aligned} \dot{x} &= Ax + Bq + Ew \\ z &= Cx + Dq + Fw \\ u &= Kx + Lq + Gw \\ q &= u - \mathrm{sat}(u) \end{aligned}$$

8. LMI Based Controller and Antiwindup Designs (Linear Controller Design)

Consider:

$$\dot{x} = Ax + Bu$$
$$u = Kx$$

Goal: Find stabilizing controller, i.e., find K and $P > 0$:

$$V(x(t)) = x(t)^T P x(t) > 0, \quad \dot{V}(x(t)) < 0 \quad \forall x(t) \neq 0$$

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In terms of definite matrices:

$$P > 0, \qquad (A + BK)^T P + P(A + BK) < 0,$$
$$P > 0, \qquad A^T P + K^T B^T P + PA + PBK < 0$$

Define $\Lambda = P^{-1}$, $\Phi = K\Lambda$:

$$\Lambda > 0, \qquad \Lambda A^T + \Lambda K^T B^T + A\Lambda + BK\Lambda < 0,$$
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LMI (as convex optimization problem):

$$\min_{\Lambda,\, \Phi} \quad f(\Lambda, \Phi)$$

subject to $\quad 0 < \ \Phi$

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Lemma (Schur Complement)

Let $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{q\times q}$, symmetric, and let $S \in \mathbb{R}^{r\times q}$. Then

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} < 0 \quad \Leftrightarrow \quad \begin{array}{r} R < 0 \\ Q - SR^{-1}S^T < 0 \end{array}$$

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Lemma (S-Lemma or S-Procedure)

Let $M_0, M_1 \in \mathbb{R}^{r\times r}$, symmetric, and suppose there exists $\zeta^* \in \mathbb{R}^r$ such that $(\zeta^*)^T M_1 \zeta^* > 0$. Then the following statements are equivalent:

1. *There exists $\tau > 0$ such that $M_0 - \tau M_1 > 0$.*
2. *For all $\zeta \neq 0$ such that $\zeta^T M_1 \zeta \geq 0$ it holds that $\zeta^T M_0 \zeta > 0$.*

- If (1) is satisfied, then (2) is satisfied
- For known τ, (1) is an LMI which can be used to verify (2).

9. Control Lyapunov Functions

Consider the nonlinear system

$$\dot{x} = f(x, u)$$

- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$
- state x and control input u
- Goal: Define a feedback control law $u = k(x)$ which asymptotically stabilizes the origin.

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Control Lyapunov function: $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$

- In terms of a feedback law $u = k(x)$,

  $$\tfrac{d}{dt} V(x(t)) = \langle \nabla V(x), f(x, k(x)) \rangle < 0, \qquad \forall\, x \neq 0$$

  $\rightsquigarrow V$ is a Lyapunov function for $\dot{x} = f(x, k(x)) = \tilde{f}(x)$

- For each $x \neq 0$ we can find u such that

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Definition (Control Lyapunov function (CLF))

Consider the nonlinear system and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called control Lyapunov function if

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \qquad \forall\, x \in \mathbb{R}^n,$$

and for all $x \in \mathbb{R}^n \backslash \{0\}$ there exists $u \in \mathbb{R}^m$ such that

$$\langle \nabla V(x), f(x,u) \rangle < 0.$$

9. Control Lyapunov Functions (Control Affine Systems)

Control affine systems

$$\dot{x} = f(x) + g(x)u$$

Assumptions:

- for simplicity we focus on $u \in \mathbb{R}$
- $f, g : \mathbb{R}^n \to \mathbb{R}^n$ (locally Lipschitz)
- $f(0) = 0$ without loss of generality

Lie derivative notation

$$L_f V(x) = \langle \nabla V(x), f(x) \rangle$$

The decrease condition:

$$\begin{aligned} \dot{V}(x) &= \langle \nabla V(x), f(x) + g(x)u \rangle \\ &= L_f V(x) + L_g V(x) u < 0, \quad \forall\, x \neq 0. \end{aligned}$$

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$$\langle \nabla V(x), f(x, u) \rangle < 0.$$

The decrease condition for control affine systems:

$$L_f V(x) < 0 \quad \forall \;\; x \in \mathbb{R}^n \backslash \{0\} \;\; \text{such that} \;\; L_g V(x) = 0$$

In other words

- If $L_g V(x) = 0$ (i.e., we have no control authority)
- then $L_f V(x) < 0$ needs to be satisfied

9. Control Lyapunov Functions (Sontag's Universal Formula)

Consider a control affine system ($u \in \mathbb{R}$)

$$\dot{x} = f(x) + g(x)u$$

with corresponding CLF V, i.e.,

$$L_f V(x) < 0 \quad \forall \ x \in \mathbb{R}^n \backslash \{0\} \ \text{such that} \ L_g V(x) = 0$$

Then, for $\kappa > 0$ define the feedback law

$$k(x) = \begin{cases} -\left(\kappa + \frac{L_f V(x) + \sqrt{L_f V(x)^2 + L_g V(x)^4}}{L_g V(x)^2}\right) L_g V(x), & L_g V(x) \neq 0 \\ 0, & L_g V(x) = 0 \end{cases}$$

The feedback law

- asymptotically stabilizes the origin
- inherits the regularity properties of the CLF except at the origin
- is continuous at the origin if the CLF satisfies a small control property (i.e., $|k(x)| \to 0$ for $|x| \to 0$)

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Sketch of the proof: For $\kappa = 0$ it holds that

$$\begin{aligned} \dot{V}(x) &= L_f V(x) + L_g V(x) k(x) \\ &= L_f V(x) - L_g V(x) \left(\frac{L_f V(x) + \sqrt{L_f V(x)^2 + L_g V(x)^4}}{L_g V(x)^2} \right) L_g V(x) \\ &= L_f V(x) - L_f V(x) - \sqrt{L_f V(x)^2 + L_g V(x)^4} = -\sqrt{L_f V(x)^2 + L_g V(x)^4}. \end{aligned}$$

- $\kappa > 0$ adds a term $-\kappa (L_g V(x))^2$ (which guarantees certain ISS properties)

Note that: Formula known as

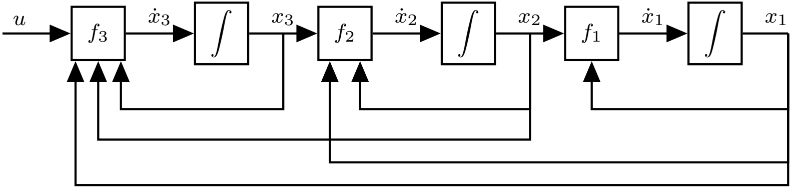
- Universal formula
- Sontag's formula

  (Derived by Eduardo Sontag)

9. Control Lyapunov Functions (Backstepping)

Systems in *strict feedback form*:

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2)\\
\dot{x}_2 &= f_2(x_1, x_2, x_3)\\
&\vdots\\
\dot{x}_{n-1} &= f_{n-1}(x_1, x_2, \ldots, x_{n-1}, x_n)\\
\dot{x}_n &= f_n(x_1, x_2, \ldots, x_n, u).
\end{aligned}
$$



10. Sliding Mode Control (Finite-Time Stability)

Consider

$$\dot{x} = f(x), \qquad x(0) = x_0 \in \mathbb{R}^n, \qquad (f(0) = 0)$$

Definition (Finite-time stability)

The origin is said to be (globally) finite-time stable if there exists a function $T : \mathbb{R}^n \backslash \{0\} \to (0, \infty)$, called the settling-time function, such that the following statements hold:

- (Stability) For every $\varepsilon > 0$ there exists a $\delta > 0$ such that, for every $x(0) = x_0 \in \mathcal{B}_\delta \backslash \{0\}$, $x(t) \in \mathcal{B}_\varepsilon$ for all $t \in [0, T(x_0))$.
- (Finite-time convergence) For every $x(0) = x_0 \in \mathbb{R}^n \backslash \{0\}$, $x(\cdot)$ is defined on $[0, T(x_0))$, $x(t) \in \mathbb{R}^n \backslash \{0\}$ for all $t \in [0, T(x_0))$, and $x(t) \to 0$ for $t \to T(x_0)$.

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Example

Consider

$$\dot{x} = f(x) = -\sqrt[3]{x^2}, \qquad \text{(with } f(0) = 0\text{)}$$

Note that

- f is not Lipschitz at the origin
- uniqueness of solutions can only be guaranteed if $x(t) \neq 0$

We can verify that

$$x(t) = -\tfrac{1}{27}\left(t - 3\,\mathrm{sign}(x(0))\sqrt[3]{|x(0)|}\right)^3$$

is a solution for all $x \in \mathbb{R}$.
However, for $x(0) > 0$

$$x(t) = \begin{cases} -\frac{1}{27}\left(t - 3\sqrt[3]{|x(0)|}\right)^3 & \text{if } t \leq 3\sqrt[3]{|x(0)|} \\ 0 & \text{if } t \geq 3\sqrt[3]{|x(0)|} \end{cases}$$

is also a solution.

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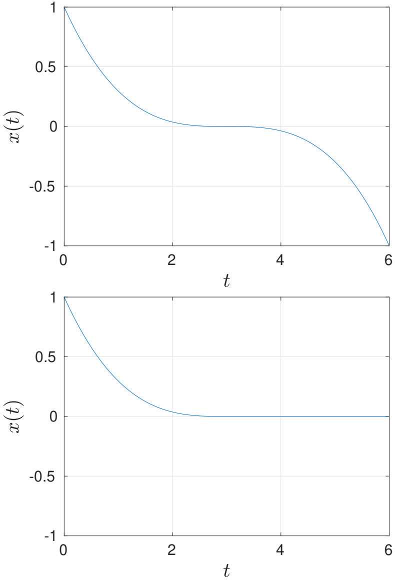

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$\rightsquigarrow$ The ODE admits unique solutions

Once the equilibrium is reached, the inequalities

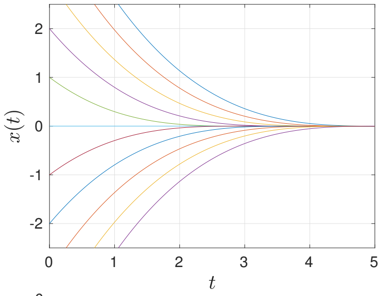
$$-\operatorname{sign}(x)\sqrt[3]{x^2} < 0 \text{ for all } x > 0, \quad \text{and}$$

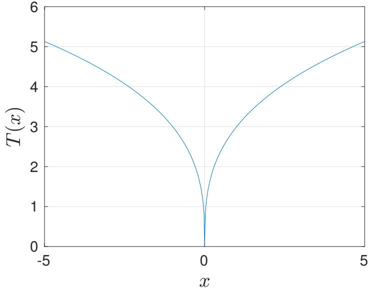
$$-\operatorname{sign}(x)\sqrt[3]{x^2} > 0 \text{ for all } x < 0$$

ensure that the origin is attractive.

It follows from the explicit solution that

- The origin is finite-time stable
- Settling time $T(x) = 3\sqrt[3]{|x|}$





10. Sliding Mode Control (Finite-Time Stability)

Theorem (Lyapunov fcn for finite-time stability)

Consider $\dot{x} = f(x)$ with $f(0) = 0$. Assume there exist a continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, which is continuously differentiable on $\mathbb{R}^n \backslash \{0\}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a constant $\kappa > 0$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|),$$

$$\dot{V}(x) = \langle \nabla V(x), f(x) \rangle \leq -\kappa \sqrt{V(x)} \qquad \forall x \neq 0.$$

Then the origin is globally finite-time stable.
Moreover, the settling-time $T(x) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is upper bounded by

$$T(x) \leq \frac{2}{\kappa} \sqrt{\alpha_2(|x|)}.$$

10. Sliding Mode Control (Example)

As an example, consider:

$$\begin{aligned} \dot{x} &= x^3 + z, \\ \dot{z} &= u + \delta(t, x, z). \end{aligned}$$

- Unknown disturbance $\delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \to \mathbb{R}$
- Assumption: there exists $L_\delta \in \mathbb{R}_{>0}$ such that

$$|\delta(t, x, z)| \leq L_\delta \qquad (t, x, z) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2$$

- Thus, δ is bounded but not necessarily continuous

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Goal: Exponential stability of the x-subsystem

- I.e., we want x to behave as $\dot{x} = -x$ (for all bounded disturbances)
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Approach: Define a new state

$$\sigma \doteq x^3 + z + x \quad \text{and} \quad V(\sigma) = \tfrac{1}{2}\sigma^2$$

- Then

$$\begin{aligned}\dot{V}(\sigma) &= \sigma\dot{\sigma} = \sigma\left(3x^2\dot{x} + \dot{z} + \dot{x}\right)\\ &= \sigma\left(3x^5 + 3x^2 z + u + \delta(t, x, z) + x^3 + z\right).\end{aligned}$$

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- To cancel the known terms define

$$u = v - 3x^5 - 3x^2 z - x^3 - z$$

so that $\quad \dot{V}(\sigma) = \sigma\left(v + \delta(t, x, z)\right) \quad$ (with new input v)

- Selecting $v = -\rho\, \mathrm{sign}(\sigma)$, $\rho > 0$, provides the estimate

$$\begin{aligned}\dot{V}(\sigma) &= \sigma\left(-\rho\, \mathrm{sign}(\sigma) + \delta(t, x, z)\right) = -\rho|\sigma| + \sigma\delta(t, x, z)\\ &\leq -\rho|\sigma| + L_\delta|\sigma| = -(\rho - L_\delta)|\sigma|.\end{aligned}$$

- Finally, with $\rho = L_\delta + \frac{\kappa}{\sqrt{2}}$, $\kappa > 0$, we have

$$\dot{V}(\sigma) \leq -\frac{\kappa|\sigma|}{\sqrt{2}} = -\alpha\sqrt{V(\sigma)} \rightsquigarrow \text{finite-time stab. of } \sigma = 0$$

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- Note that the control

$$u = -\left(L_\delta + \tfrac{\kappa}{\sqrt{2}}\right) \operatorname{sign}\left(x^3 + z + x\right) - 3x^5 - 3x^2 z - x^3 - z$$

is independent of the term $\delta(t, x, z)$.

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Control law:

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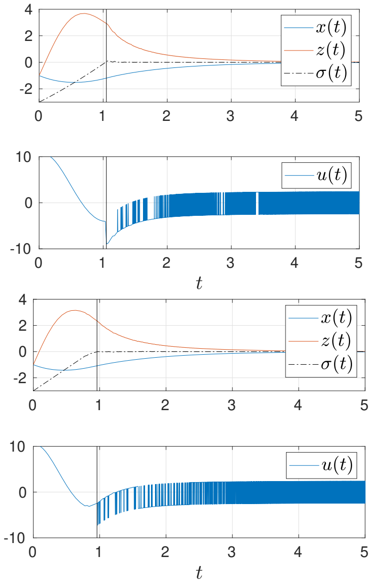
Parameter selection for the simulations:

- $L_\delta = 1$ and $\kappa = 2$
- $\delta(t, x, z) = \sin(t)$ (top)
- $\delta(t, x, z) = \operatorname{sign}(\cos(2t)\sin(2t))$ (bottom)

We observe that

- σ converges to zero in finite-time
- Afterwards (x, z) asymptotically approach the origin
- Since the ordinary differential equation is solved numerically, σ is not exactly zero!



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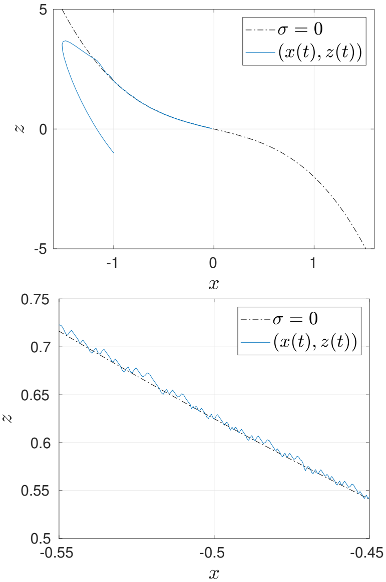

11. Adaptive Control (Motivations and Examples)

Consider parameter-dependent systems:

$$\dot{x} = f(x, u, \theta), \qquad (\theta \in \mathbb{R}^q \text{ constant but unknown})$$

Goal: Stabilization of the origin.

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Simple motivating example:

$$\dot{x} = \theta x + u$$

- Linear controller: For $u = -kx$ it holds that

  $$\dot{x} = -(k - \theta)x$$

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- Nonlinear controller: $u = -k_1 x - k_2 x^3$, $k_1, k_2 \in \mathbb{R}_{>0}$,

  $$\dot{x} = (\theta - k_1)x - k_2 x^3 = \left[(\theta - k_1) - k_2 x^2\right] x. \quad (3)$$

  - For $\theta \leq k_1$, (3) exhibits a unique equilibrium $x^e = 0$ in $\mathbb{R}$
  - For $\theta > k_1$, (3) exhibits three equilibria $x^e \in \{0, \pm\sqrt{\frac{\theta - k_1}{k_2}}\}$

⇝ It can be shown that

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- Dynamic controller: $u = -k_1 x - \xi x$, $\dot{\xi} = x^2$

$$\begin{bmatrix} \dot{x} \\ \hline \dot{\xi} \end{bmatrix} = \begin{bmatrix} \theta x - k_1 x - \xi x \\ \hline x^2 \end{bmatrix},$$

- In terms of error dynamics: $\hat{\theta} = \xi - \theta$

$$\begin{bmatrix} \dot{x} \\ \hline \dot{\hat{\theta}} \end{bmatrix} = \begin{bmatrix} -\hat{\theta} x - k_1 x \\ \hline x^2 \end{bmatrix},$$

- Lyapunov function $V(x, \hat{\theta}) = \frac{1}{2}x^2 + \frac{1}{2}\hat{\theta}^2$;

$$\dot{V}(x, \hat{\theta}) = (-(\xi - \theta)x - k_1 x)x + (\xi - \theta)x^2 = -k_1 x^2$$

⇝ $x(t) \to 0$ for $t \to \infty$ $\forall\, x(0) \in \mathbb{R}$, $\xi(0) \in \mathbb{R}$ (LaSalle-Yoshizawa theorem)

- $\xi(t) \to \theta$ for $t \to \infty$ is not guaranteed

11. Adaptive Control (Model Reference Adaptive Control)

- Consider linear systems

$$\dot{x} = Ax + Bu$$

with unknown matrices A, B.

- Goal: Design a controller so that the unknown system behaves like

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u^e$$

where $\bar{A} \in \mathbb{R}^{n\times n}$ and $\bar{B} \in \mathbb{R}^{n\times m}$ are design parameters and $u^e \in \mathbb{R}^m$ is a constant reference.

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- For $\bar{A}$ Hurwitz, u^e defines the asymptotically stable equilibrium

$$\bar{x}^e = -\bar{A}^{-1}\bar{B}u^e$$

- Control law:

$$u = M(\theta)u^e + L(\theta)x,$$

parameter dependent matrices $M(\cdot)$, $L(\cdot)$, to be designed

- Closed-loop dynamics:

$$\begin{aligned}\dot{x} &= Ax + B(M(\theta)u^e + L(\theta)x)\\ &= (A + BL(\theta))x + BM(\theta)u^e\\ &= A_{\text{cl}}(\theta)x + B_{\text{cl}}(\theta)u^e\end{aligned}$$

where

$$A_{\text{cl}}(\theta) = A + BL(\theta), \qquad B_{\text{cl}}(\theta) = BM(\theta)$$

- Compatibility conditions

$$\begin{aligned}A_{\text{cl}}(\theta) = \bar{A} \quad &\Longleftrightarrow \quad BL(\theta) = \bar{A} - A,\\ B_{\text{cl}}(\theta) = \bar{B} \quad &\Longleftrightarrow \quad BM(\theta) = \bar{B}.\end{aligned}$$

- Overall system dynamics

$$\begin{bmatrix} \dot{x} \\ \hline \dot{\bar{x}} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} (A + BL(\theta))x + BM(\theta)u^e \\ \hline \bar{A}\bar{x} + \bar{B}u^e \\ \Psi(x, \bar{x}, u^e) \end{bmatrix}$$

for Ψ defined appropriately

11. Adaptive Control (Adaptive Backstepping)

Systems in *parametric strict-feedback form*:

$$\begin{aligned}\dot{x}_1 &= x_2 + \phi_1(x_1)^T\theta \\ \dot{x}_2 &= x_3 + \phi_2(x_1, x_2)^T\theta \\ &\vdots \\ \dot{x}_{n-1} &= x_n + \phi_{n-1}(x_1, \dots, x_{n-1})^T\theta \\ \dot{x}_n &= \beta(x)u + \phi_n(x)^T\theta\end{aligned}$$

where $\beta(x) \neq 0$ for all $x \in \mathbb{R}^n$

Theorem

Let $c_i > 0$ *for* $i \in \{1, \dots, n\}$. *Consider the adaptive controller*

$$u = \tfrac{1}{\beta(x)}\alpha_n(x, \vartheta_1, \dots, \vartheta_n)$$

$$\dot{\vartheta}_i = \Gamma\left(\phi_i(x_1, \dots, x_i) - \textstyle\sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}\phi_j(x_1, \dots x_j)\right) z_i, \quad i = 1, \dots, n,$$

where $\vartheta_i \in \mathbb{R}^q$ *are multiple estimates of* θ, $\Gamma > 0$ *is the adaptation gain matrix, and the variables* z_i *and the stabilizing functions*

$$\alpha_i = \alpha_i(x_1, \dots, x_i, \vartheta_1, \dots, \vartheta_i), \qquad \alpha_i : \mathbb{R}^{i+i\cdot q} \to \mathbb{R}, \qquad i = 1, \dots, n,$$

are defined by the following recursive expressions (and $z_0 \equiv 0$, $\alpha_0 \equiv 0$ *for notational convenience)*

$$z_i = x_i - \alpha_{i-1}(x_1, \dots, x_i, \vartheta_1, \dots, \vartheta_i)$$

$$\begin{aligned}\alpha_i = &-c_i z_i - z_{i-1} - \left(\phi_i - \textstyle\sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}\phi_j\right)^T \vartheta_i \\ &+ \textstyle\sum_{j=1}^{i-1}\left(\frac{\partial \alpha_{i-1}}{\partial x_j}x_{j+1} + \frac{\partial \alpha_{i-1}}{\partial \vartheta_j}\Gamma\left(\phi_j - \sum_{k=1}^{j-1}\frac{\partial \alpha_{j-1}}{\partial x_k}\phi_k\right) z_j\right).\end{aligned}$$

This adaptive controller guarantees global boundedness of $x(\cdot)$, $\vartheta_1(\cdot)$, $\dots$, $\vartheta_n(\cdot)$, *and* $x_1(t) \to 0$, $x_i(t) \to x_i^e$ *for* $i = 2, \dots, n$ *for* $t \to \infty$ *where*

$$x_i^e = -\theta^T\phi_{i-1}(0, x_2^e, \dots, x_{i-1}^e), \qquad i = 2, \dots, n.$$

12. Optimal Control (Definitions)

We consider continuous time system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \in \mathbb{R}^n \qquad (4)$$

By assumption

- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ locally Lipschitz continuous

Set of inputs and set of solutions:

$$\mathbb{U} = \{u(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}^m |\ u(\cdot) \text{ measurable}\}$$
$$\mathbb{X} = \{x(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n |\ x(\cdot) \text{ is absolutely continuous}\}$$

We say that

- $(x(\cdot), u(\cdot)) \in \mathbb{X} \times \mathbb{U}$ is a *solution pair* if it satisfies (4) for almost all $t \in \mathbb{R}_{\geq 0}$.

Note that:

- If the initial condition is important (or not clear from context), we use $x(\cdot; x_0) \in \mathbb{X}$ and $u(\cdot; x_0) \in \mathbb{U}$
- x_0, and $u(\cdot)$ are sufficient to describe $x(\cdot)$

For $(x(\cdot), u(\cdot)) \in \mathbb{X} \times \mathbb{U}$ we define

- *Cost functional* (or performance criterion) $J : \mathbb{R}^n \times \mathbb{U} \to \mathbb{R} \cup \{\pm\infty\}$ as

$$J(x_0, u(\cdot)) = \int_0^\infty \ell(x(\tau), u(\tau)) d\tau.$$

- *Running cost*: $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$
- *(Optimal) Value function*: $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$,

$$V(x_0) = \min_{u(\cdot) \in \mathbb{U}} J(x_0, u(\cdot)) \quad \text{subject to (4).}$$

  (We assume that the minimum exists!)
- Optimal input:

$$u^\star(\cdot) = \arg \min_{u(\cdot) \in \mathbb{U}} J(x_0, u(\cdot)) \quad \text{subject to (4).}$$

12. Optimal Control (Linear Quadratic Regulator)

Linear system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n$$

Quadratic cost function:

$$J(x_0, u(\cdot)) = \int_0^\infty \left(x^T(\tau) Q x(\tau) + u^T(\tau) R u(\tau) \right) d\tau$$

Theorem

Let $Q \geq 0$, $R > 0$. If there exists $P > 0$ satisfying the continuous time algebraic Riccati equation

$$A^T P + PA + Q - PBR^{-1}B^T P = 0$$

and if $A - BR^{-1}B^T P$ is a Hurwitz matrix, then

$$\mu(x) = -R^{-1}B^T P x$$

minimizes the quadratic cost function and the optimal value function is given by

$$V(x_0) = x_0^T P x_0.$$

Linear system

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0 \in \mathbb{R}^n$$

Quadratic cost function:

$$J(x_0, u(\cdot)) = \sum_{k=0}^\infty x(k)^T Q x(k) + u(k)^T R u(k)$$

Theorem

Let $Q \geq 0$, $R > 0$. If there exists $P > 0$ satisfying the discrete time algebraic Riccati equation

$$Q + A^T PA - P - A^T PB \left(R + B^T PB \right)^{-1} B^T PA = 0$$

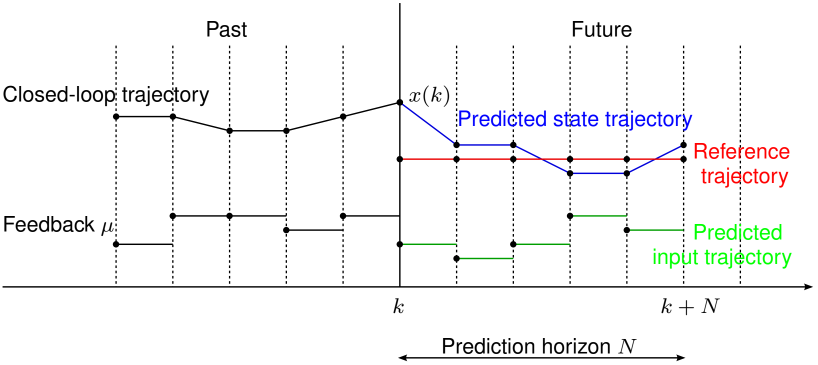
and if $A - B(R + B^T PB)^{-1} B^T PA$ is a Schur matrix, then

$$\mu(x) = -(R + B^T PB)^{-1} B^T PAx$$

minimizes the quadratic cost function and the optimal value function is given by

$$V(x_0) = x_0^T P x_0.$$

13. Model Predictive Control (Receding Horizon Principle)



MPC is also known as

- *predictive control*
- *receding horizon control*
- *rolling horizon control*

Here, we consider discrete time systems

$$x^+ = f(x,u), \qquad x(0) = x_0 \in \mathbb{R}^n$$

with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ $f(0,0) = 0$.

- State constraints $x \in \mathbb{X} \subset \mathbb{R}^n$
- Input constraints $u \in \mathbb{U}(x) \subset \mathbb{R}^m$

13. Model Predictive Control (The Basic MPC Formulation)

- Prediction horizon: $N \in \mathbb{N} \cup \{\infty\}$
- Set of feasible input trajectories of length N (depending on x_0):

$$\mathbb{U}^N_{x_0} = \left\{ u_N(\cdot) : \mathbb{N}_{[0,N-1]} \to \mathbb{R}^m \;\middle|\; \begin{array}{rcl} x(0) & = & x_0, \\ x(k+1) & = & f(x(k), u(k)) \\ (x(k), u(k)) & \in & \mathbb{X} \times \mathbb{U}(x) \\ & \forall & k \in \mathbb{N}_{[0,N-1]} \end{array} \right\}$$

- For clarity, note that

$$u_N(\cdot; x_0) = u_N(\cdot) = [u_N(0), u_N(1), u(2), \ldots, u_N(N-1)]$$

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- For clarity, note that

$$u_N(\cdot; x_0) = u_N(\cdot) = [u_N(0), u_N(1), u(2), \ldots, u_N(N-1)]$$

- *Cost function*: $J_N : \mathbb{R}^n \times \mathbb{U}_{\mathbb{D}}^N \to \mathbb{R} \cup \{\infty\}$,

$$J_N(x_0, u_N(\cdot)) = \textstyle\sum_{i=0}^{N-1} \ell(x(i), u(i))$$

(with running costs $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$)

- *Terminal cost* $F : \mathbb{R}^n \to \mathbb{R}$ and *terminal constraints* $\mathbb{X}_F \subset \mathbb{R}^n$

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- *Terminal cost* $F : \mathbb{R}^n \to \mathbb{R}$ and *terminal constraints* $\mathbb{X}_F \subset \mathbb{R}^n$
- Optimal control problem

$$V_N(x_0) = \min_{u_N(\cdot) \in \mathbb{U}_{x_0}^N} J_N(x_0, u_N(\cdot)) + F(x(N))$$

$$\text{subject to dyn. \& init. cond. and } x(N) \in \mathbb{X}_F$$

($\rightsquigarrow$ finite dimensional optimization problem if N is finite)

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($\rightsquigarrow$ finite dimensional optimization problem if N is finite)

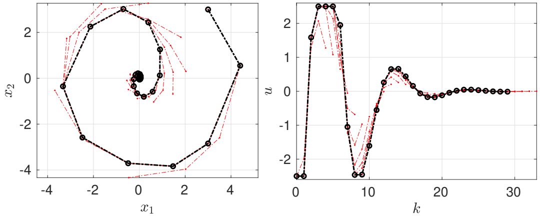
- Even if $V_N : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is not known explicitly, for a given $x_0 \in \mathbb{R}^n$, the function $V_N(\cdot)$ can be evaluated in x_0 by solving the OCP.
- *Optimal open-loop input trajectory* $u_N^\star(\cdot; x_0) \in \mathbb{U}_{\mathbb{D}}^N$ s.t. $x(N) \in \mathbb{X}_F$ &

$$V_N(x_0) = J_N(x_0, u_N^\star(\cdot; x_0)) + F(x(N))$$

- $u_N^\star(\cdot; x_0)$ is used to iteratively define a *feedback law* μ_N, i.e.,

$$\mu_N(x_0) = u_N^\star(0; x_0)$$
$$x_{\mu_N}(k+1) = f(x_{\mu_N}(k), \mu_N(x(k)))$$

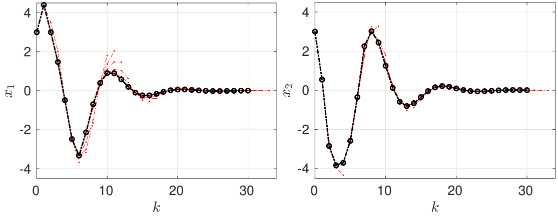
13. Model Predictive Control (Example)



Consider $x^+ = Ax + Bu$ with unstable origin and

$$A = \begin{bmatrix} \frac{6}{5} & \frac{6}{5} \\ -\frac{1}{2} & \frac{6}{5} \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

- Prediction horizon: $N = 5$
- The running cost: $\ell(x,u) = x^T x + 5u^2$
- Constraints: $u \in \mathbb{U} = [-2.5, 2.5]$, $x \in \mathbb{R}^2$ (i.e., $\mathbb{D} = \mathbb{R}^2 \times \mathbb{U}$)
- Terminal cost & constraints: $F(x) = x^T x$, $\mathbb{X}_F = \mathbb{R}^2$.



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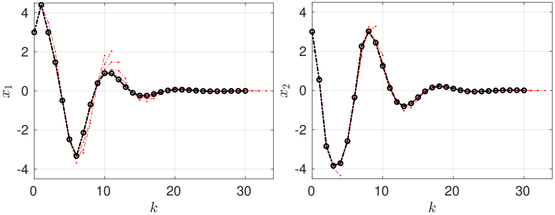
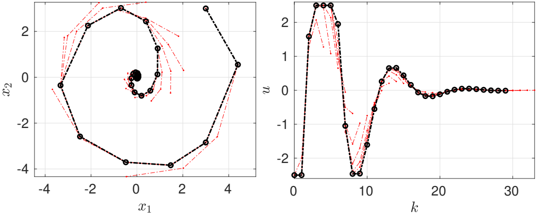
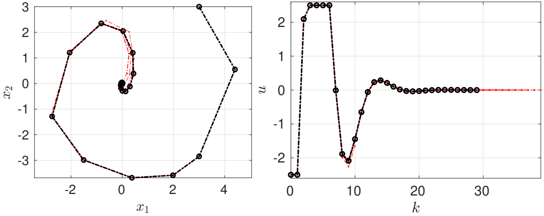




- Now, use the terminal constraint $\mathbb{X}_F = \{0\}$ (which makes $F(x)$ superfluous)
- Prediction horizon $N = 11$ (since for $N < 11$ the OCP is not feasible for $x_0 = [3\ 3]^T$)



A Run Through Nonlinear Control Topics

Stability, control design, and estimation

Philipp Braun

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CIICADA

Australian National University

3. Linear Systems (Controllability & Observability)

Linear system with output:

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

Definition (Controllability)

The linear system (or (A, B)) is said to be controllable, if for all $x_1, x_2 \in \mathbb{R}^n$ there exists $T \in \mathbb{R}_{\geq 0}$ and $u : [0, T] \to \mathbb{R}^m$ such that

$$x_2 = e^{AT} x_1 + \int_0^T e^{A(T-\tau)} Bu(\tau) d\tau.$$

Ability of a system to steer any initial state to a target state through an appropriate input $u : [0, T] \to \mathbb{R}^m$.

Theorem (Controllability, Kalman matrix)

Consider the linear system defined through the pair (A, B). *The linear system (or equivalently the pair* (A, B)*) is controllable if and only if*

$$\operatorname{rank} \left([B \ AB \ A^2B \ \cdots \ A^{n-1}B] \right) = n.$$

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Ability of a system to steer any initial state to a target state through an appropriate input $u : [0, T] \to \mathbb{R}^m$.

Definition (Observability)

The linear system (or (A, C)) is said to be observable, if for all $x_1, x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$ there exists $T \in \mathbb{R}_{\geq 0}$ such that

$$Ce^{AT} x_2 \neq Ce^{AT} x_1.$$

Determines if $x(0)$ can be uniquely determined by measuring $y(t) = Cx(t)$ over a given time window $t \in [0, T]$.

Theorem (Controllability, Kalman matrix)

Consider the linear system defined through the pair (A, B). The linear system (or equivalently the pair (A, B)) is controllable if and only if

$$\mathrm{rank}\left(\left[B \; AB \; A^2B \; \cdots \; A^{n-1}B\right]\right) = n.$$

Theorem (Observability)

Consider the linear system defined through the pair (A, C). The linear system with output (or equivalently the pair (A, C)) is observable if and only if

$$\mathrm{rank}\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n.$$

- (A, B) controllable if and only if (A^T, B^T) observable
- (A, C) observable if and only if (A^T, C^T) controllable

4. Frequency Domain Analysis (The transfer function)

Consider single-input single-output (SISO) linear systems:

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Rearrange the terms ($x(0) = 0$):

$$\hat{y}(s) = \left(c(sI - A)^{-1}b + d\right)\hat{u}(s)$$

Identify input output relationship:

$$G(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = c(sI - A)^{-1}b + d \qquad (5)$$

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Definition (Realization)

Consider a transfer function $G(s)$ and assume that (5) is satisfied for (A, b, c, d). Then $G(s)$ is called realizable and the quadruple (A, b, c, d) is called a realization of $G(s)$.

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Theorem (Minimal realization)

The quadruple (A, b, c, d) is a minimal realization of $G(s) = c(sI - A)^{-1}b + d$ if and only if (A, b) is controllable and (A, c) is observable.

Theorem (Uncontrollable & unobs. modes)

Let (A, b, c, d) be a minimal realization of $G(s) = \frac{P(s)}{Q(s)}$. Then $\lambda \in \mathbb{C}$ is a pole of G, i.e., $Q(\lambda) = 0$, if and only if λ is an eigenvalue of A.

Definition (BIBO stability)

The linear system is called bounded-input, bounded-output (BIBO) stable if $\|u\|_{\mathcal{L}_\infty} < \infty$ implies $\|y\|_{\mathcal{L}_\infty} < \infty$.

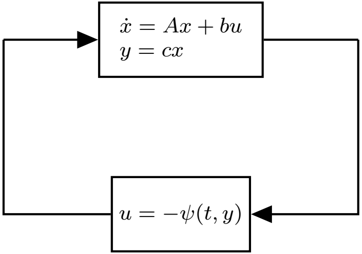
Graphical tools:

- The Bode Plot & The Nyquist Criterion

$\mathcal{L}_\infty$-norm: $\|\psi\|_{\mathcal{L}_\infty[0,t)} = \operatorname{ess\,sup}_{\tau\in[0,t)} |\psi(\tau)| = \inf\{\eta \in \mathbb{R}_{\geq 0} : |\psi(t)| \leq \eta$ for almost all $\tau \in [0,t)\}$

6. Absolute Stability (The Lur'e Problem)

Consider the feedback interconnection:



Lur'e problem:

- Which conditions on the functions $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ guarantee asymptotic stability of the origin?

Note that:

- The nonlinearity can be time-dependent
- We assume that the reference signal $v(t)$ is zero.
- While we focus on the SISO case, many results can be extended to the MIMO case.

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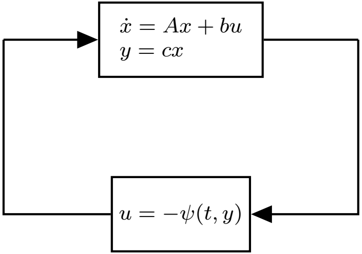

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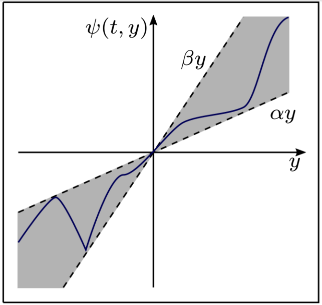
- While we focus on the SISO case, many results can be extended to the MIMO case.

Definition (Sector condition)

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and $\Omega \subset \mathbb{R}$. A nonlinearity $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ satisfies a sector condition if

$$\alpha y^2 \leq y\psi(t, y) \leq \beta y^2$$

for all $t \geq 0$ and for all $y \in \Omega$. For $\Omega = \mathbb{R}$ we say that the sector condition is satisfied globally.



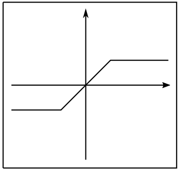
6. Absolute Stability (Sector Condition)

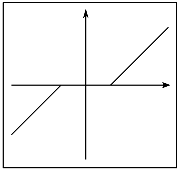
Common nonlinearities: $\mathrm{sign} : \mathbb{R} \to \mathbb{R}$,

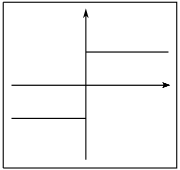
$$\mathrm{sat}(y) = \begin{cases} -1, & \text{for } y \leq -1, \\ y, & \text{for } -1 \leq y \leq 1, \\ 1, & \text{for } y \geq 1. \end{cases}$$

$$\mathrm{dz}(y) = \begin{cases} y+1, & \text{for } y \leq -1, \\ 0, & \text{for } -1 \leq y \leq 1, \\ y-1, & \text{for } y \geq 1. \end{cases}$$

$$\mathrm{sign}(y) = \begin{cases} -1, & \text{for } y < 0, \\ 0, & \text{for } y = 0, \\ 1, & \text{for } y > 0, \end{cases}$$







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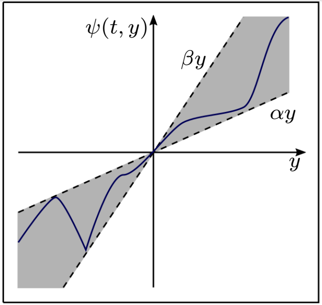

Question:

- Which nonlinearity satisfies a sector condition?

6. Absolute Stability (Definition & Conjectures)

Definition (Sector condition)

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for all $t \geq 0$ and for all $y \in \Omega$. For $\Omega = \mathbb{R}$ we say that the sector condition is satisfied globally.

Definition (Absolute stability)

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and $\Omega \subset \mathbb{R}$. The Lur'e system

$$\dot{x} = Ax - b\psi(t, y)$$

is called absolutely stable (with respect to α, β, Ω) if the origin is asymptotically stable for all $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ satisfying the sector condition for all $t \geq 0$ and for all $y_0 \in \Omega$.

6. Absolute Stability (Definition & Conjectures)

Definition (Sector condition)

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and $\Omega \subset \mathbb{R}$. A nonlinearity $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ satisfies a sector condition if

$$\alpha y^2 \leq y\psi(t, y) \leq \beta y^2$$

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Conjecture (Aizerman's Conjecture (1949))

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and suppose the origin of the linear system $\dot{x} = Ax + bu$, $y = cx$ is globally asymptotically stable for all linear feedbacks

$$u = -\psi(y) = -ky, \quad k \in [\alpha, \beta].$$

Then the origin is globally asymptotically stable for all nonlinear feedbacks in the sector

$$\alpha \leq \frac{\psi(y)}{y} \leq \beta, \quad y \neq 0.$$

$\rightsquigarrow$ Conjecture was shown to be wrong through counterexamples.

6. Absolute Stability (Definition & Conjectures)

Definition (Sector condition)

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and $\Omega \subset \mathbb{R}$. A nonlinearity $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ satisfies a sector condition if

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Definition (Absolute stability)

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and $\Omega \subset \mathbb{R}$. The Lur'e system

$$\dot{x} = Ax - b\psi(t, y)$$

is called absolutely stable (with respect to α, β, Ω) if the origin is asymptotically stable for all $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ satisfying the sector condition for all $t \geq 0$ and for all $y_0 \in \Omega$.

Conjecture (Kalman's Conjecture (1957))

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and suppose the origin of the linear system $\dot{x} = Ax + bu$, $y = cx$ is globally asymptotically stable for all linear feedbacks

$$u = -\psi(y) = -ky, \quad k \in [\alpha, \beta].$$

Then the origin is globally asymptotically stable for all nonlinear feedbacks belonging to the incremental sector

$$\alpha \leq \frac{\partial}{\partial y}\psi(y) \leq \beta.$$

$\rightsquigarrow$ Conjecture was shown to be wrong through counterexamples.

6. Absolute Stability (Prepratation; Circle Criterion)

Definitions: (Disc in the complex plane)

- center $\sigma : \mathbb{R}\backslash\{0\} \times \mathbb{R}_{>0} \to \mathbb{R}$
- radius $r : \mathbb{R}\backslash\{0\} \times \mathbb{R}_{>0} \to \mathbb{R}$
- for $\alpha \neq 0$ and $\beta > 0$ we define

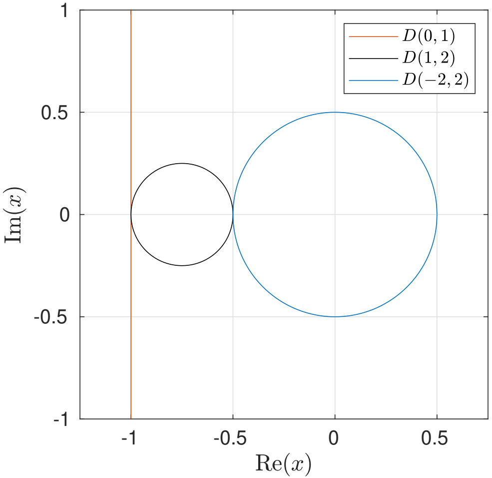
$$\sigma(\alpha, \beta) = \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right), \quad r(\alpha, \beta) = \frac{\text{sign}(\alpha)}{2}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)$$

Then, the disc $D(\cdot, \cdot)$ is defined as

$$D(\alpha, \beta) = \begin{cases} \{x \in \mathbb{C} : x = -\frac{1}{\beta} + j\omega, \omega \in \mathbb{R}\}, & \text{if } \alpha = 0 < \beta, \\ \{x \in \mathbb{C} : |x - \sigma(\alpha, \beta)| = r(\alpha, \beta)\}, & \text{if } 0 < \alpha < \beta, \\ \{x \in \mathbb{C} : |x - \sigma(\alpha, \beta)| = r(\alpha, \beta)\}, & \text{if } \alpha < 0 < \beta. \end{cases}$$

Note that

- for $\alpha \neq 0$, $D(\alpha, \beta)$ defines a disc centered around $\sigma(\alpha, \beta)$ with radius $r(\alpha, \beta)$
- for $\alpha = 0$, $D(0, \beta)$ defines a vertical line



6. Absolute Stability (Circle Criterion)

Theorem (Circle Criterion)

Suppose (A, b, c) *is a minimal realization of* $G(s)$ *and* $\psi(t, y)$ *satisfies the sector condition*

$$\alpha y^2 \leq y\psi(t, y) \leq \beta y^2$$

globally. Then the system is absolutely stable if:

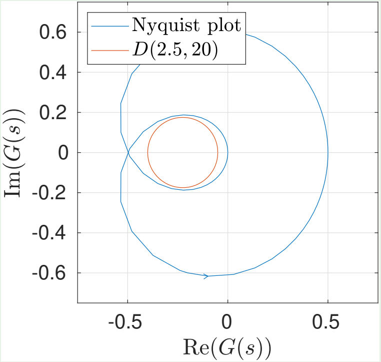
1. $\alpha = 0 < \beta$, *the Nyquist plot is to the right of the line* $\text{Re}(s) = -\frac{1}{\beta}$, *(i.e., to the right of* $D(0, \beta)$*) and* $G(s)$ *is Hurwitz;*
2. $0 < \alpha < \beta$, *the Nyquist plot does not enter the disk* $D(\alpha, \beta)$, *and encircles it in the counter-clockwise direction as many times,* N, *as there are right-half plane poles of* $G(s)$*; or*
3. $\alpha < 0 < \beta$, *the Nyquist plot lies in the interior of the disk* $D(\alpha, \beta)$, *and* $G(s)$ *is Hurwitz.*

Example

Consider the transfer function

$$G(s) = \frac{s+1}{s^2 - 2s + 2} = \frac{s+1}{(s-1+j)(s-1-j)}$$

Two poles in right-half plane $\rightsquigarrow$ absolute stability (Item 2)



13. Model Predictive Control (Algorithm)

Input: Measurement of the initial condition $x(0)$; prediction horizon $N \in \mathbb{N} \cup \{\infty\}$; running cost $\ell : \mathbb{R}^{n+m} \to \mathbb{R}$; constraints $\mathbb{D} \subset \mathbb{R}^{n+m}$; terminal cost $F : \mathbb{R}^n \to \mathbb{R}$ and terminal constraints $\mathbb{X}_F \subset \mathbb{R}^n$.

For $k = 0, 1, 2, \ldots$

1. Measure the current state of the system $x^+ = f(x, u)$ and define $x_0 = x(k)$.
2. Solve the optimal control problem

$$V_N(x_0) = \min_{u_N(\cdot) \in \mathbb{U}_{\mathbb{D}}^N} J_N(x_0, u_N(\cdot)) + F(x(N))$$

$$\text{subject to dyn. \& init. cond. and } x(N) \in \mathbb{X}_F$$

   to obtain the open-loop input $u_N^\star(\cdot; x_0)$.

3. Define the feedback law

$$\mu_N(x(k)) = u_N^\star(0; x_0).$$

4. Compute $x(k+1) = f(x(k), \mu_N(x(k)))$, increment k to $k + 1$ and go to 1.

14. Differential Geometric Methods

Consider:

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, $f(0) = 0$.
Goal: Compute coordinate transformation

$$z = \Phi(x), \qquad \Phi : \mathbb{R}^n \to \mathbb{R}^n$$

so that

$$\begin{bmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_{r-1} \\ \dot{z}_r \end{bmatrix} = \begin{bmatrix} z_2 \\ \vdots \\ z_r \\ \alpha(z) + \beta(z)u \end{bmatrix}$$

$$\begin{bmatrix} \dot{z}_{r+1} \\ \vdots \\ \dot{z}_n \end{bmatrix} = \gamma(z)$$

$$y = z_1$$

where $r \in \{1, \ldots, n\}$ and $\alpha, \beta : \mathbb{R}^n \to \mathbb{R}$, $\gamma : \mathbb{R}^n \to \mathbb{R}^{n-r}$.

If Φ is known and $\beta(z) \neq 0$, then:

- the coordinate transformation $v = \alpha(z) + \beta(z)u$ leads to a linear controller (in v) can be used to ensure $y(t) \to 0$
- the control law

$$u = \frac{1}{\beta(\Phi(x))}\left(v - \alpha(\Phi(x))\right)$$

  in the original variables is only well-defined if $z_{r+1}, \ldots, z_n$ are well behaved.

$\rightsquigarrow$ Feedback Linearization

Coordinate transformation leads to

- input-to-state linearization (if $r = n$)
- input-to-output linearization (if $r < n$)

14. Differential Geometric Methods (Relative degree and coordinate transformation)

Consider:

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

Lie derivative: $f, g, h : \mathbb{R}^n \to \mathbb{R}$

$$L_f\lambda(x) = \langle \nabla\lambda(x), f(x) \rangle$$

Repeated Lie derivatives:

$$L_f^0 h(x) = h(x)$$
$$L_g L_f h(x) = \langle \nabla L_f h(x), g(x) \rangle,$$
$$L_f^k h(x) = \langle \nabla L_f^{k-1} h(x), f(x) \rangle$$

Definition (Relative degree)

The system has *relative degree* $r \in \mathbb{N}$ at a point $x^\circ \in \mathbb{R}^n$ if

(i) the repeated Lie derivatives satisfy $L_g L_f^k h(x) = 0$ for all x in a neighborhood of x° and all $k < r - 1$; and

(ii) the repeated Lie derivative satisfies $L_g L_f^{r-1} h(x^\circ) \neq 0$.

Remark

The relative degree of a linear system $y(s) = \frac{P(s)}{Q(s)} u(s)$ is defined as the difference between the degree of the denominator and numerator.

Coordinate transformation:

- For $r = n$, define

$$z = \Phi(x) = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_r(x) \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{r-1} h(x) \end{bmatrix}. \qquad (6)$$

- If $r \neq n$, augment (6) with additional $n - r$ functions.

Consider:

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

Lie derivative: $f, g, h : \mathbb{R}^n \to \mathbb{R}$

$$L_f \lambda(x) = \langle \nabla \lambda(x), f(x) \rangle$$

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$$L_f^k h(x) = \langle \nabla L_f^{k-1} h(x), f(x) \rangle$$

Proposition

Consider the system with relative degree $r \in \mathbb{N}$ at $x^\circ \in \mathbb{R}^n$.

- *If $r < n$, then there exist $n - r$ functions $\phi_{r+1}, \ldots, \phi_n : \mathbb{R}^n \to \mathbb{R}$, so that $\Phi(x) = [\phi_1, \ldots, \phi_n]^T$ has a nonsingular Jacobian at x^0 and*

$$L_g \phi_i(x) = 0, \quad r + 1 \leq i \leq n.$$

- *For $r \leq n$, the coordinate transformation satisfies*

$$\dot{z}_1 = \langle \nabla \phi_1(x), \dot{x} \rangle = L_f h(x) + L_g h(x) u = L_f h(x) = z_2$$
$$\dot{z}_2 = \langle \nabla (L_f h(x)), \dot{x} \rangle = L_f^2 h(x) = z_3$$
$$\vdots$$
$$\dot{z}_{r-1} = \langle \nabla (L_f^{r-2} h(x)), \dot{x} \rangle = L_f^{r-1} h(x) = z_r$$
$$\dot{z}_r = \langle \nabla (L_f^{r-1} h(x)), \dot{x} \rangle = L_f^r h(x) + L_g L_f^{r-1} h(x) u,$$

and if $r < n$, the remaining coordinates $i \in \{r+1, \ldots, n\}$ satisfy

$$\dot{z}_i = \langle \nabla \phi_i(x), \dot{x} \rangle = L_f \phi_i(x) + L_g \phi_i(x) u = L_f \phi_i(x).$$

14. Differential Geometric Methods (Input-to-state & input-to-output linearization)

Consider:

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

Lie derivative: $f, g, h : \mathbb{R}^n \to \mathbb{R}$

$$L_f\lambda(x) = \langle \nabla\lambda(x), f(x)\rangle$$

Repeated Lie derivatives:

$$L_f^0 h(x) = h(x)$$
$$L_g L_f h(x) = \langle \nabla L_f h(x), g(x)\rangle,$$
$$L_f^k h(x) = \langle \nabla L_f^{k-1} h(x), f(x)\rangle$$

Additional remarks:

- Lie bracket: $f, g : \mathbb{R}^n \to \mathbb{R}^n$,

$$[f, g](x) = \tfrac{\partial g}{\partial x}(x) f(x) - \tfrac{\partial f}{\partial x}(x) g(x)$$

⇝ Concept used to verify controllability of nonlinear systems

- The zero dynamics are the internal dynamics when the output is kept at 0 by u

Proposition

Consider the system with relative degree $r \in \mathbb{N}$ at $x^\circ \in \mathbb{R}^n$.

- *If $r < n$, then there exist $n - r$ functions $\phi_{r+1}, \ldots, \phi_n : \mathbb{R}^n \to \mathbb{R}$, so that $\Phi(x) = [\phi_1, \ldots, \phi_n]^T$ has a nonsingular Jacobian at x^0 and*

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$$\dot{z}_{r-1} = \langle \nabla(L_f^{r-2} h(x)), \dot{x}\rangle = L_f^{r-1} h(x) = z_r$$
$$\dot{z}_r = \langle \nabla(L_f^{r-1} h(x)), \dot{x}\rangle = L_f^r h(x) + L_g L_f^{r-1} h(x) u,$$

and if $r < n$, the remaining coordinates $i \in \{r+1, \ldots, n\}$ satisfy

$$\dot{z}_i = \langle \nabla\phi_i(x), \dot{x}\rangle = L_f\phi_i(x) + L_g\phi_i(x) u = L_f\phi_i(x).$$