Introduction to Nonlinear Control

Stability, control design, and estimation

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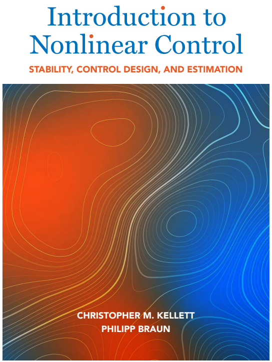

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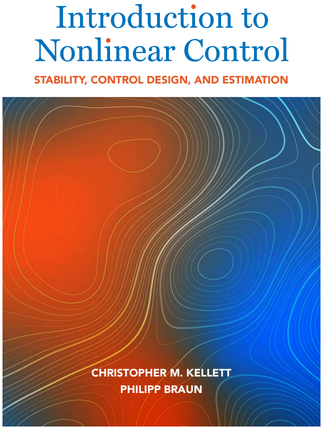
Part I: Dynamical Systems



State Space Models

First order differential equations :
(or time-invariant system or autonomous system)

$$\dot{x}(t) = \frac{d}{dt}x(t) = f(x(t)), \quad f : \mathbb{R}^n \to \mathbb{R}^n \qquad (1)$$

- A *solution* of (1) is an absolutely continuous function that satisfies (1) for almost all t.
- If f is (locally) Lipschitz, then there exists $\delta > 0$ so that (1) has a unique solution over $[t_0, t_0 + \delta]$.
- Short-hand notation: $\dot{x} = f(x)$

Non-autonomous/time-varying system:

$$\dot{x}(t) = f(t, x(t)), \qquad f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n \qquad (2)$$

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Systems with external inputs $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$:

$$\dot{x} = f(x, u), \qquad \dot{x} = f(x, w),$$

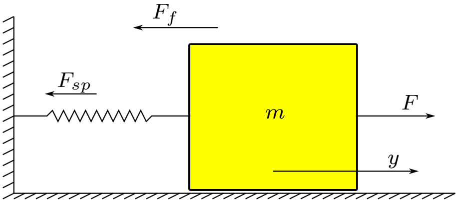
- $u : \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto u(x)$ $\leftarrow$ degree of freedom (input)
- $w : \mathbb{R} \to \mathbb{R}^m$, $t \mapsto w(t)$ $\leftarrow$ exogenous signal (disturbance or reference)

Systems with output:

$$\begin{aligned} \dot{x} &= f(x, u), & f &: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \\ y &= h(x, u), & h &: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p \end{aligned}$$

- State: $x \in \mathbb{R}^n$
- Input: $u \in \mathbb{R}^m$
- Output: $y \in \mathbb{R}^p$

State Space Models (Example: Mass-Spring System)



Mass m, restoring force of the spring F_{sp}, friction force F_f, external driving force F, displacement y.

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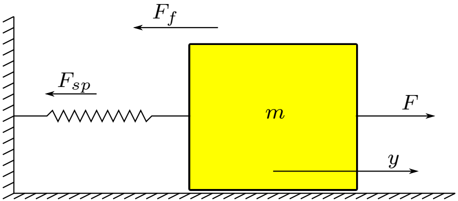

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Newton's second law of motion:

$$m\ddot{y} = F - F_f - F_{sp} = F - c\dot{y} - ky \qquad (3)$$

- Viscous friction: $F_f = c\dot{y}$
- Linear spring: $F_{sp} = ky$
- Input: $F = u$

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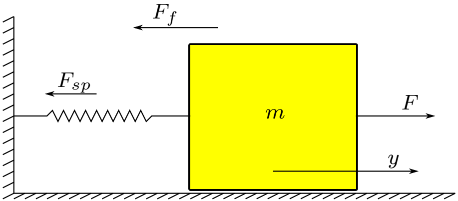

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From second order to first order dynamics:

- Coordinate transformation

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \begin{array}{l} x_1 = y \\ x_2 = \dot{y} \end{array} \qquad \Longrightarrow \qquad \begin{array}{l} \dot{x}_1 = \dot{y} \\ \dot{x}_2 = \ddot{y} \end{array}$$

- then

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}u$$

(Linear) Dynamical systems:

$$\dot{x} = f(x,u) = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}u \end{bmatrix}$$
$$y = h(x,u) = x_1$$

General Question:

- (Depending on the input u) How does the position $y(t)$ evolve over time?

Equilibria and pairs of induced equilibria

Definition (Equilibrium, $\dot{x} = 0$)

$x^e \in \mathbb{R}^n$ is called an equilibrium of $\dot{x} = f(x)$ or $\dot{x} = f(t, x)$, respectively, if

$$\dot{x} = f(x^e) = 0,$$
$$\dot{x} = f(t, x^e) = 0 \qquad \forall t \in \mathbb{R}_{\geq 0}.$$

The pair $(x^e, u^e) \in \mathbb{R}^n \times \mathbb{R}^m$ is called an equilibrium pair of the system $\dot{x} = f(x, u)$ if

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- Without loss of generality $x^e = 0$ (or $(x^e, u^e) = 0$).
- To see this, consider coordinate transf. $z = x - x^e$.
- Then

$$\tfrac{d}{dt} z(t) = \tfrac{d}{dt} x(t) - \tfrac{d}{dt} x^e = f(x(t)) = f(z(t) + x^e).$$

and

$$\hat{f}(z) \doteq f(z + x^e) \qquad \text{yields} \qquad \dot{z} = \hat{f}(z)$$

where $(z^e = 0)$

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Recall the mass-spring system:

$$0 \overset{!}{=} \dot{x}_1 = x_2$$
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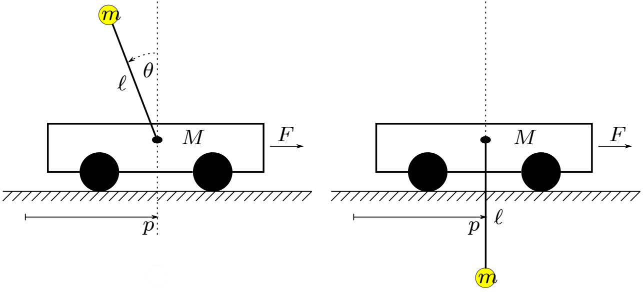
In the case that $u = 0$:

- The first equation implies that $x_2 = 0$.
- The second equation implies that $x_1 = 0$.
- Equilibrium: $x_1 = y = 0$, $x_2 = \dot{y} = 0$.

Exercise: How do equilibrium pairs look like?

Examples of dynamical systems: The inverted pendulum on a cart



General dynamics of a mechanical system:

$$M(q)\ddot{q} + C(q,\dot{q}) + K(q) = B(q)u$$

- $M(q)$: inertia matrix
- $C(q,\dot{q})$: Coriolis forces
- $K(q)$: potential energy terms
- $B(q)$: external forces

$$\begin{bmatrix} M+m & -ml\cos(\theta) \\ -ml\cos(\theta) & J+ml^2 \end{bmatrix} \begin{bmatrix} \ddot{p} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} c\dot{p} + ml\sin(\theta)\dot{\theta}^2 \\ \gamma\dot{\theta} - mgl\sin(\theta) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

$$q = \begin{bmatrix} p \\ \theta \end{bmatrix},$$ parameters, states, inputs

Exercises:

- Rewrite the system as $\dot{x} = f(x,u)$
- For $F = 0$ compute equilibria of $\dot{x} = f(x,u)$

Comparison Functions (as tools in modern control theory)

Definition (Class-$\mathcal{P}, \mathcal{K}, \mathcal{K}_\infty, \mathcal{L}, \mathcal{KL}$ functions)

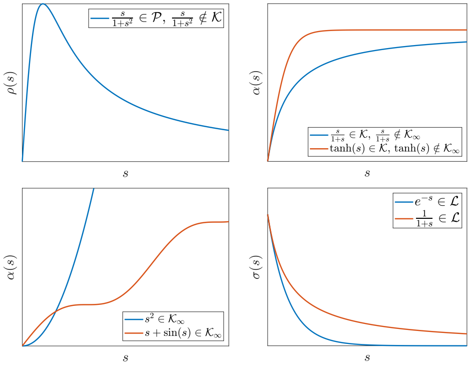
- A continuous function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a positive definite ($\rho \in \mathcal{P}$) if $\rho(0) = 0$ and $\rho(s) > 0 \; \forall \; s \in \mathbb{R}_{>0}$.
- $\alpha \in \mathcal{P}$ is said to be of class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if α strictly increasing.
- $\alpha \in \mathcal{K}$ is said to be of class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if $\lim_{s \to \infty} \alpha(s) = \infty$.
- A continuous function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class-$\mathcal{L}$ ($\sigma \in \mathcal{L}$) if σ is strictly decreasing and $\lim_{s \to \infty} \sigma(s) = 0$.
- A continuous function $\beta : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class-$\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if for each fixed $t \in \mathbb{R}_{\geq 0}$, $\beta(\cdot, t) \in \mathcal{K}_\infty$ and for each fixed $s \in \mathbb{R}_{>0}$, $\beta(s, \cdot) \in \mathcal{L}$.

Some properties:

- If $\alpha_1 \in \mathcal{K}_\infty$ then $\alpha_2 = \alpha_1^{-1} \in \mathcal{K}_\infty$
- If $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ then

$$\alpha(s) \doteq \alpha_1(\alpha_2(s)) = \alpha_1 \circ \alpha_2(s) \in \mathcal{K}_\infty.$$

- If $\alpha \in \mathcal{K}$, $\sigma \in \mathcal{L}$ then $\alpha \circ \sigma \in \mathcal{L}$.



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