

Introduction to Nonlinear Control

Stability, control design, and estimation

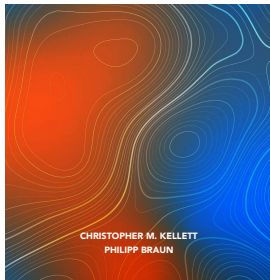
Christopher M. Kellett & Philipp Braun



Introduction to Nonlinear Control

STABILITY, CONTROL DESIGN, AND ESTIMATION

CHRISTOPHER M. KELLETT
PHILIPP BRAUN



Part I: Dynamical Systems

1. Nonlinear Systems - Fundamentals & Examples

1.1 State Space Models

1.1.1 Notational Conventions

1.1.2 Rescaling

1.1.3 Comparison Functions

1.2 Control Loops, Controller Design & Examples

1.2.1 The Pendulum on a Cart

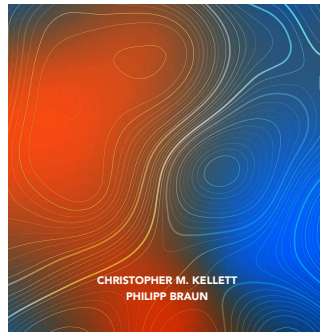
1.2.2 Mobile Robots – The Nonholonomic Integrator

1.3 Exercises

1.4 Bibliographical Notes and Further Reading

Introduction to Nonlinear Control

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State Space Models

First order differential equations :
(or time-invariant system or autonomous system)

$$\dot{x}(t) = \frac{d}{dt}x(t) = f(x(t)), \quad f : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad (1)$$

- A **solution** of (1) is an **absolutely continuous function** that satisfies (1) for almost all t .
- If f is (locally) Lipschitz, then there exists $\delta > 0$ so that (1) has a **unique solution** over $[t_0, t_0 + \delta]$.
- Short-hand notation: $\dot{x} = f(x)$

Non-autonomous/time-varying system:

$$\dot{x}(t) = f(t, x(t)), \quad f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \quad (2)$$

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Systems with external inputs $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$:

$$\dot{x} = f(x, u), \quad \dot{x} = f(x, w),$$

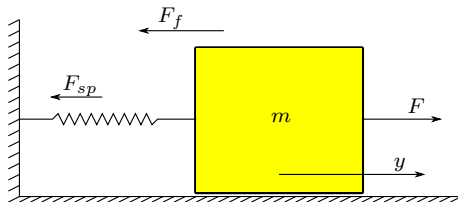
- $u : \mathbb{R}^n \rightarrow \mathbb{R}^m, x \mapsto u(x)$ **← degree of freedom (input)**
- $w : \mathbb{R} \rightarrow \mathbb{R}^m, t \mapsto w(t)$ **← exogenous signal (disturbance or reference)**

Systems with output:

$$\begin{aligned} \dot{x} &= f(x, u), & f : \mathbb{R}^n \times \mathbb{R}^m &\rightarrow \mathbb{R}^n \\ y &= h(x, u), & h : \mathbb{R}^n \times \mathbb{R}^m &\rightarrow \mathbb{R}^p \end{aligned}$$

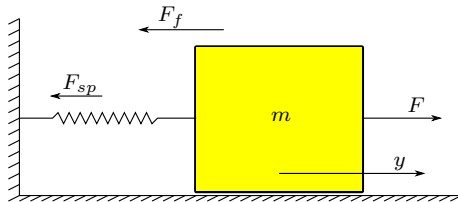
- State: $x \in \mathbb{R}^n$
- Input: $u \in \mathbb{R}^m$
- Output: $y \in \mathbb{R}^p$

State Space Models (Example: Mass-Spring System)



Mass m , restoring force of the spring F_{sp} , friction force F_f , external driving force F , displacement y .

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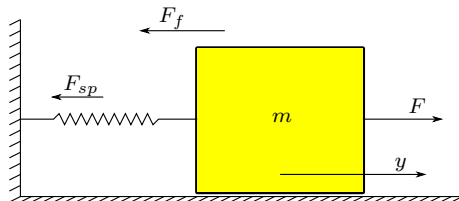
Mass m , restoring force of the spring F_{sp} , friction force F_f , external driving force F , displacement y .

Newton's second law of motion:

$$m\ddot{y} = F - F_f - F_{sp} = F - c\dot{y} - ky \quad (3)$$

- Viscous friction: $F_f = c\dot{y}$
- Linear spring: $F_{sp} = ky$
- Input: $F = u$

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From second order to first order dynamics:

- Coordinate transformation

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \begin{matrix} x_1 = y \\ x_2 = \dot{y} \end{matrix} \quad \Rightarrow \quad \begin{matrix} \dot{x}_1 = \dot{y} \\ \dot{x}_2 = \ddot{y} \end{matrix}$$

- then

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}u$$

(Linear) Dynamical systems:

$$\dot{x} = f(x, u) = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}u \end{bmatrix}$$
$$y = h(x, u) = x_1$$

General Question:

- (Depending on the input u) How does the position $y(t)$ evolve over time?

Equilibria and pairs of induced equilibria

Definition (Equilibrium, $\dot{x} = 0$)

$x^e \in \mathbb{R}^n$ is called an **equilibrium** of $\dot{x} = f(x)$ or $\dot{x} = f(t, x)$, respectively, if

$$\dot{x} = f(x^e) = 0,$$

$$\dot{x} = f(t, x^e) = 0 \quad \forall t \in \mathbb{R}_{\geq 0}.$$

The pair $(x^e, u^e) \in \mathbb{R}^n \times \mathbb{R}^m$ is called an **equilibrium pair** of the system $\dot{x} = f(x, u)$ if

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- Without loss of generality $x^e = 0$ (or $(x^e, u^e) = 0$).
- To see this, consider **coordinate transf.** $z = x - x^e$.
- Then

$$\frac{d}{dt}z(t) = \frac{d}{dt}x(t) - \frac{d}{dt}x^e = f(x(t)) = f(z(t) + x^e).$$

and

$$\hat{f}(z) \doteq f(z + x^e) \quad \text{yields} \quad \dot{z} = \hat{f}(z)$$

where ($z^e = 0$)

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Recall the mass-spring system:

$$0 \stackrel{!}{=} \dot{x}_1 = x_2$$

$$0 \stackrel{!}{=} \dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}u$$

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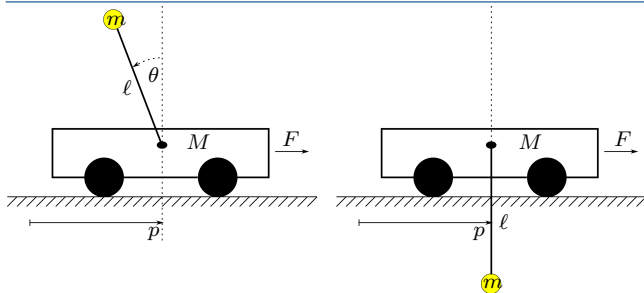
$$0 \stackrel{!}{=} \dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}u$$

In the case that $u = 0$:

- The first equation implies that $x_2 = 0$.
- The second equation implies that $x_1 = 0$.
- **Equilibrium:** $x_1 = y = 0$, $x_2 = \dot{y} = 0$.

Exercise: How do equilibrium pairs look like?

Examples of dynamical systems: The inverted pendulum on a cart



General dynamics of a mechanical system:

$$M(q)\ddot{q} + C(q, \dot{q}) + K(q) = B(q)u$$

- $M(q)$: inertia matrix
- $C(q, \dot{q})$: Coriolis forces
- $K(q)$: potential energy terms
- $B(q)$: external forces

$$\begin{bmatrix} M + m & -ml \cos(\theta) \\ -ml \cos(\theta) & J + ml^2 \end{bmatrix} \begin{bmatrix} \ddot{p} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} c\dot{p} + ml \sin(\theta)\dot{\theta}^2 \\ \gamma\dot{\theta} - mgl \sin(\theta) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

$$q = \begin{bmatrix} p \\ \theta \end{bmatrix}, \quad \text{parameters, states, inputs}$$

Exercises:

- Rewrite the system as $\dot{x} = f(x, u)$
- For $F = 0$ compute equilibria of $\dot{x} = f(x, u)$

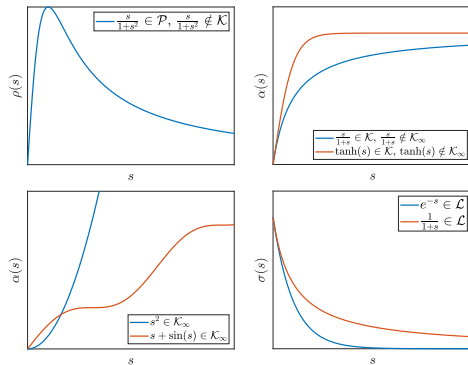
Comparison Functions (as tools in modern control theory)

Definition (Class- \mathcal{P} , \mathcal{K} , \mathcal{K}_∞ , \mathcal{L} , \mathcal{KL} functions)

- A continuous function $\rho : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be a **positive definite** ($\rho \in \mathcal{P}$) if $\rho(0) = 0$ and $\rho(s) > 0 \forall s \in \mathbb{R}_{>0}$.
- $\alpha \in \mathcal{P}$ is said to be of **class- \mathcal{K}** ($\alpha \in \mathcal{K}$) if α strictly increasing.
- $\alpha \in \mathcal{K}$ is said to be of **class- \mathcal{K}_∞** ($\alpha \in \mathcal{K}_\infty$) if $\lim_{s \rightarrow \infty} \alpha(s) = \infty$.
- A continuous function $\sigma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be of **class- \mathcal{L}** ($\sigma \in \mathcal{L}$) if σ is strictly decreasing and $\lim_{s \rightarrow \infty} \sigma(s) = 0$.
- A continuous function $\beta : \mathbb{R}_{\geq 0}^2 \rightarrow \mathbb{R}_{\geq 0}$ is said to be of **class- \mathcal{KL}** ($\beta \in \mathcal{KL}$) if for each fixed $t \in \mathbb{R}_{\geq 0}$, $\beta(\cdot, t) \in \mathcal{K}_\infty$ and for each fixed $s \in \mathbb{R}_{>0}$, $\beta(s, \cdot) \in \mathcal{L}$.

Some properties:

- If $\alpha_1 \in \mathcal{K}_\infty$ then $\alpha_2 = \alpha_1^{-1} \in \mathcal{K}_\infty$
- If $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ then $\alpha(s) \doteq \alpha_1(\alpha_2(s)) = \alpha_1 \circ \alpha_2(s) \in \mathcal{K}_\infty$.
- If $\alpha \in \mathcal{K}$, $\sigma \in \mathcal{L}$ then $\alpha \circ \sigma \in \mathcal{L}$.



Introduction to Nonlinear Control: Stability, control design, and estimation

Part I: Dynamical Systems

1. Nonlinear Systems - Fundamentals & Examples
2. Nonlinear Systems - Stability Notions
3. Linear Systems and Linearization
4. Frequency Domain Analysis
5. Discrete Time Systems
6. Absolute Stability
7. Input-to-State Stability

Part II: Controller Design

8. LMI Based Controller and Antiwindup Designs
9. Control Lyapunov Functions
10. Sliding Mode Control
11. Adaptive Control
12. Introduction to Differential Geometric Methods
13. Output Regulation
14. Optimal Control
15. Model Predictive Control

Part III: Observer Design & Estimation

16. Observer Design for Linear Systems
17. Extended & Unscented Kalman Filter & Moving Horizon Estimation
18. Observer Design for Nonlinear Systems